DIGITAL SIGNAL PROCESSING

UNIT-I
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Discrete Time Fourier Transform

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Concept of Signal and Classification:

Signal can be defined as a function or any physical phenomenon that conveys or carries some information and its amplitude may vary with respect to one or more independent variables.

Examples:

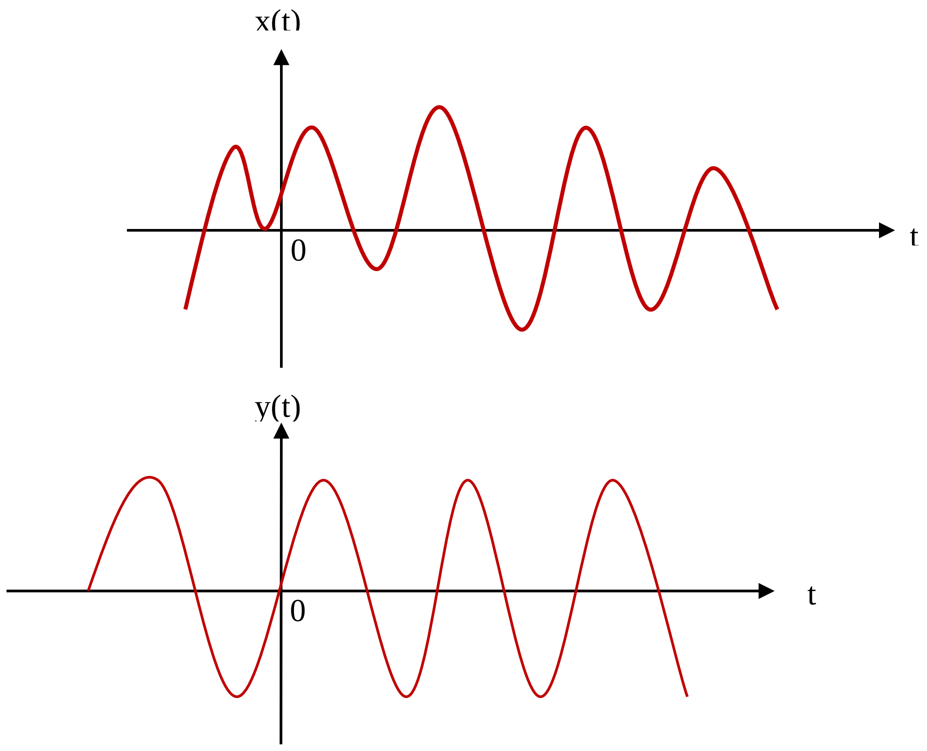
- Music Signal
- Speech Signal
- Video Signal
- Electrocardiogram (ECG) Signal – It is used to predict heart diseases
- Electroencephalogram (EEG) Signal – Study the normal and abnormal behavior of the brain
- Electromyography (EMG) – It is used to study the condition of muscles
- Electromagnetic Waves
- Radar Signals

Based on variation in amplitude, signals are classified into mainly two types.

- Continuous Time or Analog Signals
- Discrete Time and Digital Signals

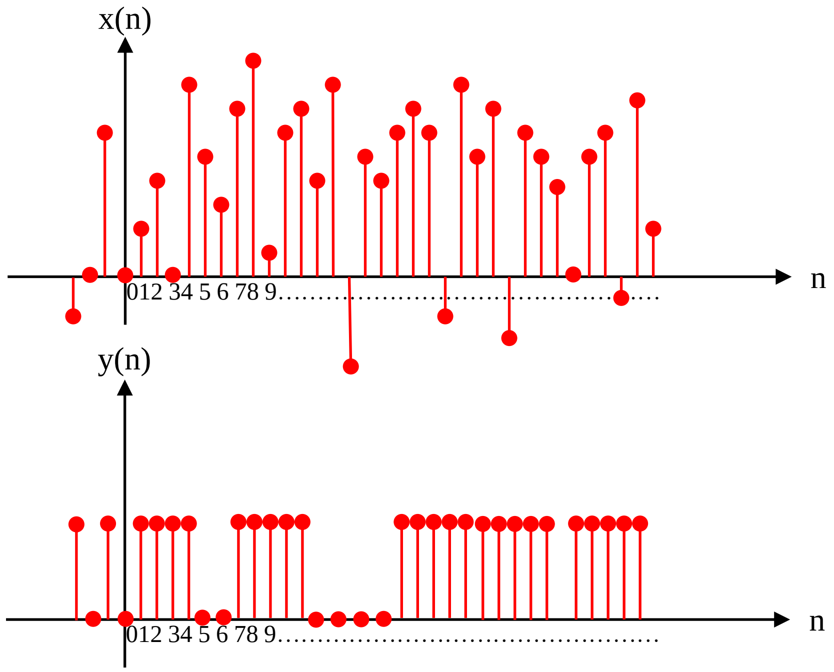
(A)Continuous Time or Analog Signals:

Continuous time signals are those for which the amplitude varies continuously in accordance with continuous variation in time. All real time signals are analog in nature, hence the continuous time signals are also known as analog signals. In general, continuous time signals are represented with x(t), y(t), z(t), z(t), etc.



(B)Discrete Time and Digital Signals:

Discrete time signals are those for which the amplitude varies discreetly in accordance with discrete variation in time. Any discrete time signal can be represented as the sequence of numbers, that's why discrete time signals are called sequences. Discrete time signals can be obtained from continuous time signals by sampling process. In general, discrete time signals are represented with x(n), y(n), z(n), etc.



Amplitude restricted version of discrete time signals are called digital signals, for which the different number of amplitudes are restricted to finite number (Two). y(n) is a digital signal and all digital signals are discrete time signals. Digital signals can be obtained from discrete time signals by quantization mechanism.

Examples:

- $x(t) = 2Cos(3t) + 3Sin(2t)$
- $y(t) = 3e^{-2t}$
- $z(t) = 3e^{-2t}Cos(4t)$

} Continuous time or analog signals

- $x(n) = 2^n$
- $y(n) = 2Cos(3n - 4)$
- $z(n) = 2e^{jn\pi/3}$

} Discrete time signals

- $x(n) = \{1,1,0,1,0,1,1,1,0,0,1\}$
- $y(n) = \{1,-1,1,-1,1,1,-1,-1,1\}$
- $z(n) = \{1,1,2,1,2,1,1,1,2,2,1\}$

} Discrete time or Digital signals

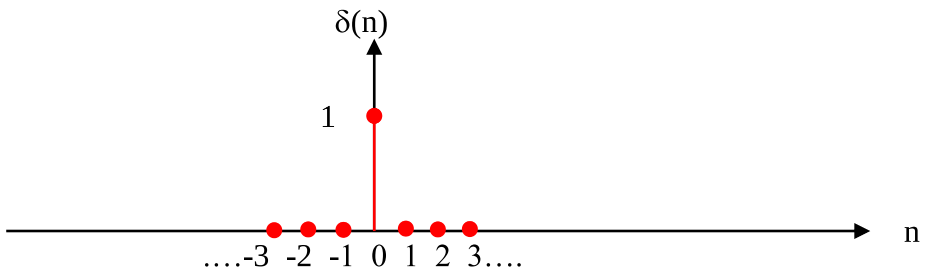
Elementary or Standard Discrete Time Signals:

Most commonly used signals are called elementary or standard discrete time signals, like digital impulse signal or unit sample sequence, unit step signal, unit ramp signal, decaying exponential signal, raising exponential signal, double exponential signal, etc.

(A)Digital Impulse Signal or Unit Sample Sequence:

Digital impulse or unit sample sequence is denoted with δ(n) and it can be defined as

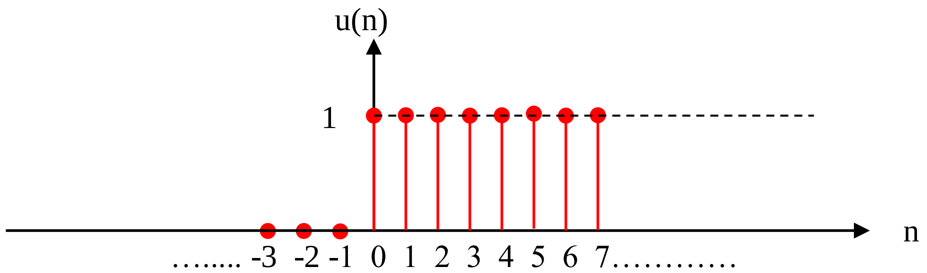
$$\delta(n) = \begin{cases} 1 & ; \ n = 0 \\ 0 & ; \ n \neq 0 \end{cases}$$



(B)Unit Step Signal:

Unit step signal is denoted with u(n) and it can be defined as

$$u(n) = \begin{cases} 1 & ; \ n \geq 0 \\ 0 & ; \ n < 0 \end{cases}$$



Unit step signal u(n) is the sum of a train of unit sample sequences

$$u(n) = \delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3) + \ldots\ldots\ldots + \delta(n-k) + \ldots\ldots$$

$$u(n) = \sum_{k=0}^{\infty} \delta(n-k) \quad \text{or} \quad u(n) = \sum_{k=-\infty}^{n} \delta(k)$$

Unit sample sequence δ(n) is difference between u(n) and u(n-1)

$$\delta(n) = u(n) - u(n-1)$$

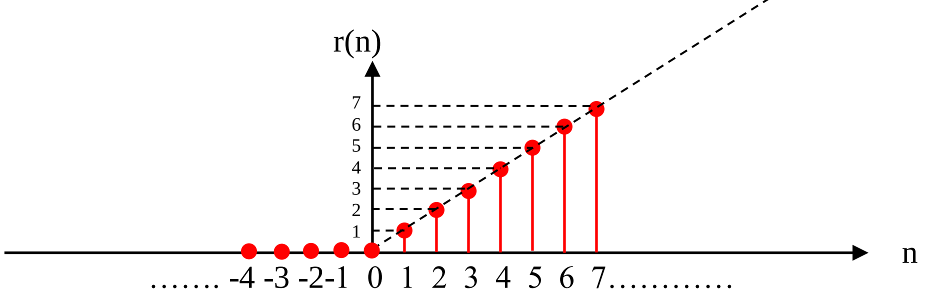
A sequence x(n) can be represented using unit sample sequence δ(n)

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\, \delta(n-k)$$

(C)Unit Ramp Signal:

Unit ramp signal is denoted with r(n) and it can be defined as

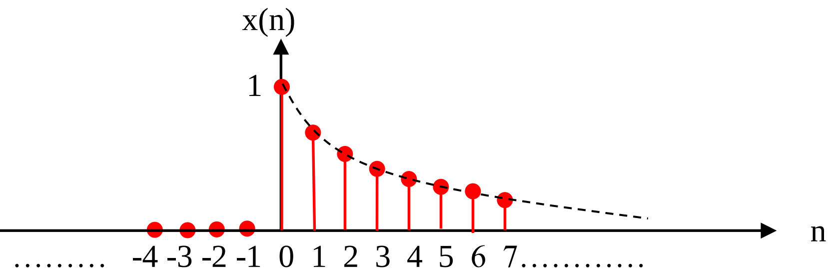
$$r(n)=nu(n)=\begin{cases} n & ;\ n\geq 0 \\ 0 & ;\ n<0 \end{cases}$$



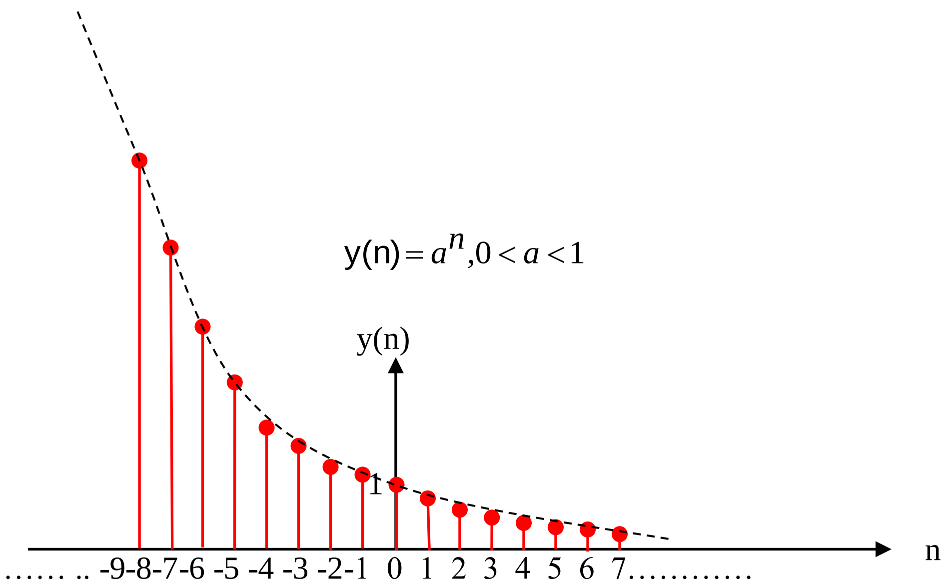
(D)Decaying Exponential Sequence:

A sequence x(n) is said to be decaying exponential only when the amplitude decays exponentially if the time increases.

$$x(n)=a^{n}u(n),0<a<1$$



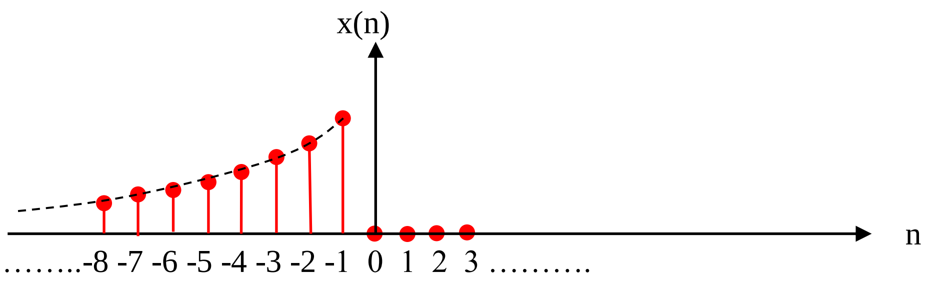
$$y(n)=a^{n},0<a<1$$



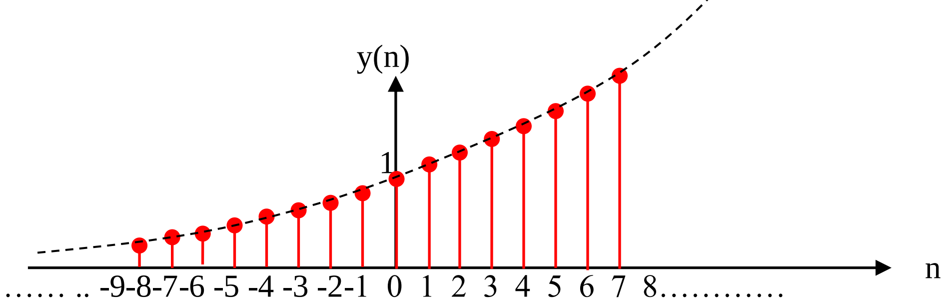
(E)Raising Exponential Sequence:

A sequence x(n) is said to be raising exponential only when the amplitude raises exponentially if the time increases.

$$x(n) = a^n u(-n-1), a > 1$$



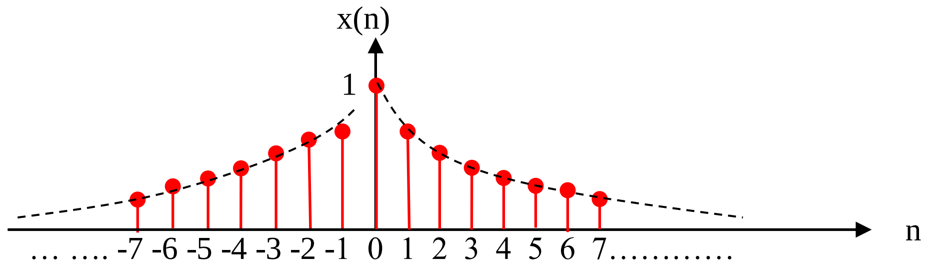
$$y(n) = a^n, a > 1$$



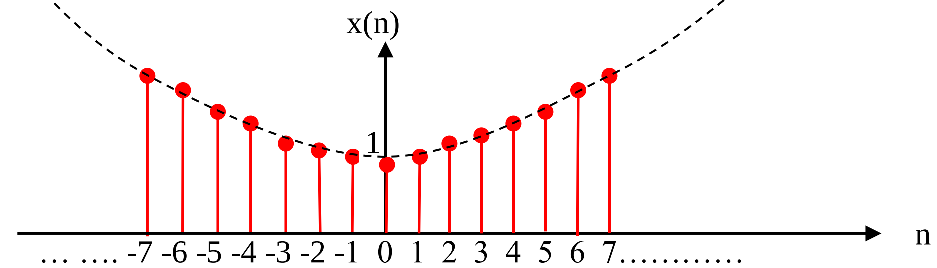
(F)Double Exponential Sequence:

A double exponential sequence x(n) is the combination of both raising and decaying exponential.

$$x(n) = a^{|n|}, 0 < a < 1$$



$$y(n) = a^{|n|}, a > 1$$



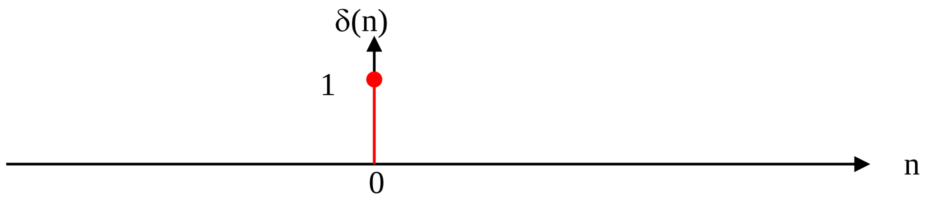
Representation of Discrete Time Signals:

A discrete time signal can be represented in four different ways

- Graphical Representation – Signal in the form of graph
- Functional Representation – Mathematical Representation
- Tabular Representation – Signal in the form of table
- Sequence Representation – Signal in the form of sequence

Example-1:

Graphical representation of unit sample sequence x(n) = δ(n)



Functional representation of unit sample sequence x(n) = δ(n)

$$\delta(n) = \begin{cases} 1 & ;\ n = 0 \\ 0 & ;\ n \neq 0 \end{cases}$$

Tabular representation of unit sample sequence x(n) = δ(n)

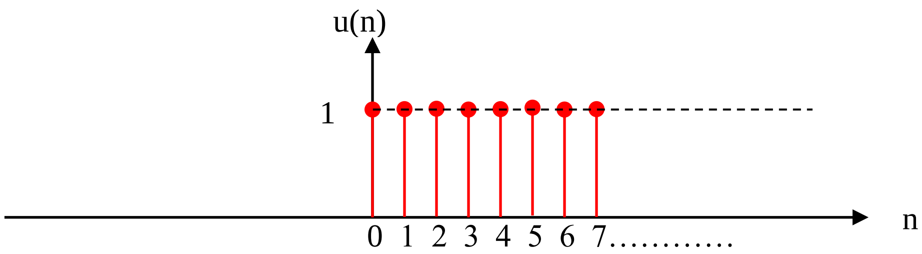
n	…	…	-3	-2	-1	0	1	2	3	…	…
δ(n)	0	0	0	0	0	1	0	0	0	0	0

Sequence representation of unit sample sequence x(n) = δ(n)

x(n) = δ(n) = {1}

Example-2:

Graphical representation of unit step sequence x(n) = u(n)



Functional representation of unit step sequence x(n) = u(n)

$$u(n) = \begin{cases} 1 & ;\ n \geq 0 \\ 0 & ;\ n < 0 \end{cases}$$

Tabular representation of unit step sequence x(n) = u(n)

n	…	…	-3	-2	-1	0	1	2	3	…	…
δ(n)	0	0	0	0	0	1	1	1	1	1	1

Sequence representation of unit step sequence x(n) = u(n)

$$x(n) = u(n) = \left\{ \underset{\uparrow}{1},1,1,1,1,1,1,1......... \right\}$$

Operations on Discrete Time Signals:

Various operations used on discrete time signals are given below

- Time Shifting
- Time Scaling
- Time Reversal or Folding
- Amplitude Scaling
- Convolution

(A)Time Shifting Operation:

If the time shifting operation is applied on a discrete time signal x(n), then the signal is shifted to left or right without changing its characteristics(width, amplitude and area). It is represented with $x(n \pm n_0)$, where n_0 is shift and it may be advance or delay.

Example-1:

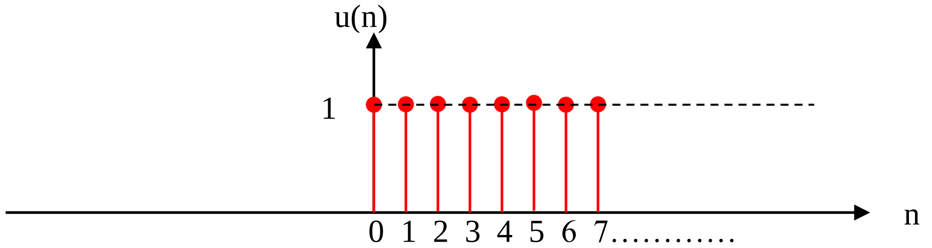
Given discrete time signal, $x(n) = \left\{6,-2,3,\underset{\uparrow}{1},4,5,-1,7\right\}$

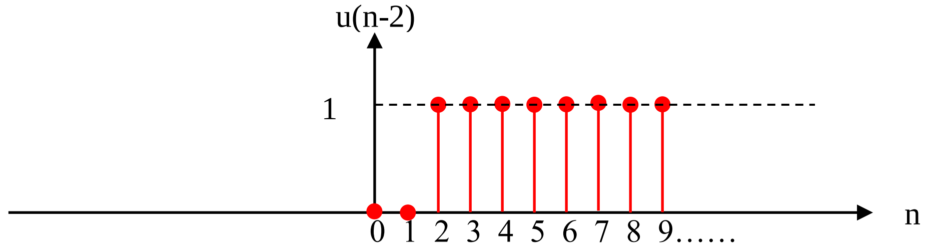
$x(n-2) = \left\{6,-\underset{\uparrow}{2},3,1,4,5,-1,7\right\}$, it is shifted to right by 2 units.

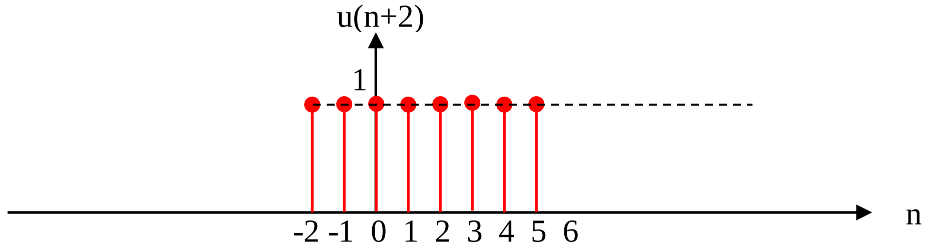
$x(n+3) = \left\{6,-2,3,1,4,5,-\underset{\uparrow}{1},7\right\}$, it is shifted to left by 3 units.

Example-2:

Given discrete time signal, x(n) = u(n)







(B)Time Scaling Operation:

If the time scaling operation is applied on a discrete time signal x(n), then the signal is compressed or expanded in time axes without changing its amplitude. It is represented with x(an), where 'a' is time scaling parameter. If a>1, then it is compressed and if 0<a<1, then it is expanded signal.

Example:

Given discrete time signal, $x(n) = \left\{1,2,3,\underset{\uparrow}{4},5,6,7,8,9\right\}$

$x(2n) = \left\{2,\underset{\uparrow}{4},6,8\right\}$, it is compressed signal.

$x(n/2) = \left\{1,0,2,0,3,0,\underset{\uparrow}{4},0,5,0,6,0,7,0,8,0,9\right\}$, it is expanded signal.

(C)Time Reversal or Folding Operation:

Time reversal signal can be obtained by interchanging left hand side and right hand side samples with respect to vertical axes or y-axes. If the given discrete time signal is x(n), then its time reversal form is represented with x(-n) and the operation is called folding or mirror image or time reversal.

Example:

Given discrete time signal, $x(n) = \left\{1,2,3,\underset{\uparrow}{4},5,6,7,8,9\right\}$

Then, time reversal signal $x(-n) = \left\{9,8,7,6,5,\underset{\uparrow}{4},3,2,1\right\}$

(D)Amplitude Scaling Operation:

If the amplitude scaling operation is applied on a discrete time signal x(n), then the signal amplitude may increase or decrease without changing its duration. It is represented with Ax(n), where 'A' is amplitude scaling parameter.

Example:

Given discrete time signal, $x(n) = \left\{1,2,3,\underset{\uparrow}{4},5,6,7,8,9\right\}$

$y(n) = 2x(n) = \left\{2,4,6,\underset{\uparrow}{8},10,12,14,16,18\right\}$

$z(n) = \frac{1}{2}x(n) = \left\{0.5,1,1.5,\underset{\uparrow}{2},2.5,3,3.5,4,4.5\right\}$

Example:

Determine (a) y(n)=3x(2n-4) (b) z(n)=3x(-2n-4). (c) y(n)+z(n). Given $x(n)=\left\{1,2,3,\underset{\uparrow}{4},5,6,7,8,9\right\}$,

Given x(0)=4, x(1)=5, x(2)=6, x(3)=6, x(4)=7, x(5)=8, x(6)=9,x(-1)=3, x(-2)=2 & x(-3)=1

$$(a)y(n)=3x(2n-4)$$

$$=\left\{3x(-6),\underset{\uparrow}{3x(-4)},3x(-2),3x(0),3x(2),3x(4),3x(6),3x(8)\right\}$$

$$=\left\{\underset{\uparrow}{0},6,12,18,21,27\right\}$$

$$(b)z(n)=3x(-2n-4)$$

$$=\left\{3x(6),3x(4),3x(2),3x(0),3x(-2),\underset{\uparrow}{3x(-4)},3x(-6)\right\}$$

$$=\left\{27,21,18,12,6,\underset{\uparrow}{0}\right\}$$

$$(c)y(n)+z(n)=\left\{27,21,18,12,6,\underset{\uparrow}{0},6,12,18,21,27\right\}$$

(E)Convolution Operation:

Convolution is an operation, which is used in almost all signal processing applications to analyze signals and systems in both the time and frequency domain. Convolution is a special operation, which includes four different operations, namely

- Folding,
- Shifting,
- Multiplication and
- Summation in the case of discrete time signals or

  Integration in the case of continues time signals.

Convolution in continuous time domain can be defined as

$$x_1(t)*x_2(t)=\int_{-\infty}^{\infty}x_1(\tau)\,x_2(t-\tau)d\tau$$

Where, $x_1(t)$ and $x_2(t)$ are two continuous time signals

Convolution in discrete time domain can be defined as

$$x_1(n)*x_2(n)=\sum_{m=-\infty}^{\infty}x_1(m)\,x_2(n-m)$$

Where, $x_1(n)$ and $x_2(n)$ are two discrete time signals

Example-1:

Compute the convoluted sequence $x(n) = x_1(n) * x_2(n)$ using

(a)Graphical method

(b)Tabular method

(c)Matrix method

Given $x_1(n) = \{1,2,3,4\}$ and $x_2(n) = \{5,6,7,8,9\}$

(a)Graphical method

- Draw the graphical representation of given sequences $x_1(m)$ and $x_2(m)$.
- Take the folding form of $x_2(m)$ to get $x_2(-m)$.
- Shift the folding sequence $x_2(-m)$ in different cases to get $x_2(n-m)$.
- Finally apply convolution formula to get the convoluted sequence.

$$x(n) = x_1(n) * x_2(n) = \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(n-m)$$

$$n = 0 \Rightarrow x(0) = \sum_{m=0}^{7} x_1(m)\, x_2(-m) = 1\times 5 = 5$$

$$n = 1 \Rightarrow x(1) = \sum_{m=0}^{7} x_1(m)\, x_2(1-m) = 1\times 6 + 2\times 5 = 6 + 10 = 16$$

$$n = 2 \Rightarrow x(2) = \sum_{m=0}^{7} x_1(m)\, x_2(2-m) = 1\times 7 + 2\times 6 + 3\times 5 = 7 + 12 + 15 = 34$$

$$n = 3 \Rightarrow x(3) = \sum_{m=0}^{7} x_1(m)\, x_2(3-m) = 1\times 8 + 2\times 7 + 3\times 6 + 4\times 5 = 8 + 14 + 18 + 20 = 60$$

$$n = 4 \Rightarrow x(4) = \sum_{m=0}^{7} x_1(m)\, x_2(4-m) = 1\times 9 + 2\times 8 + 3\times 7 + 4\times 6 = 9 + 16 + 21 + 24 = 70$$

$$n = 5 \Rightarrow x(5) = \sum_{m=0}^{7} x_1(m)\, x_2(5-m) = 2\times 9 + 3\times 8 + 4\times 7 = 18 + 24 + 28 = 70$$

$$n = 6 \Rightarrow x(6) = \sum_{m=0}^{7} x_1(m)\, x_2(6-m) = 3\times 9 + 4\times 8 = 27 + 32 = 59$$

$$n = 7 \Rightarrow x(7) = \sum_{m=0}^{7} x_1(m)\, x_2(7-m) = 4\times 9 = 36$$

Convoluted Sequence $x(n) = x_1(n) * x_2(n) = \{5, 16, 34, 60, 70, 70, 59, 36\}$

Given Sequences:

Folding:

Shifting:

Case –1:

Case –2:

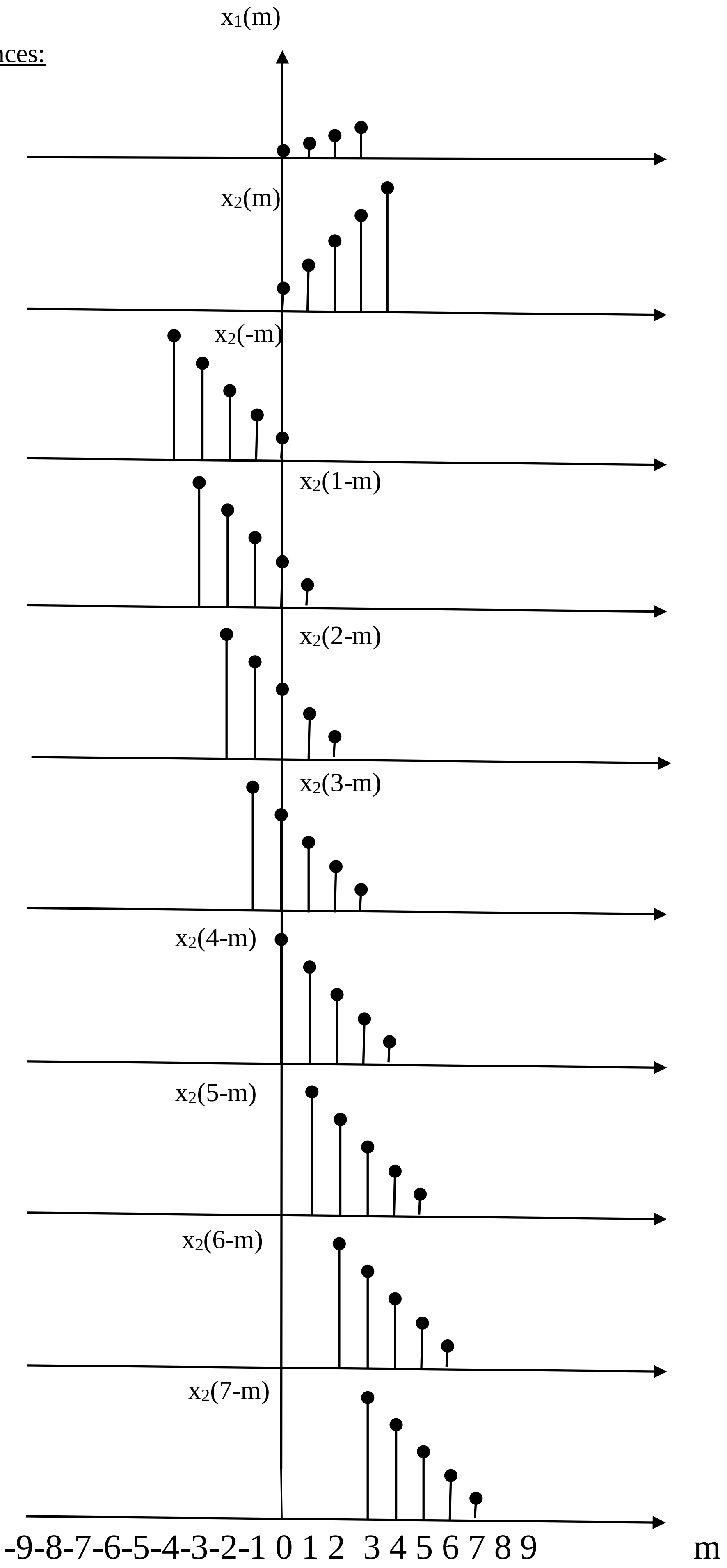
Case –3:

Case –4:

Case –5:

Case –6:

Case –7:



(b)Tabular method:

- Draw a table and represent the samples of $x_1(m)$ and $x_2(m)$ in a table**.**
- Take the folding form of $x_2(m)$ and represent the samples of $x_2(-m)$ in a table
- Shift the folding sequence $x_2(-m)$ in different cases and represent the samples in a table
- Apply multiplication operation to get the samples of $x_1(m)\, x_2(n-m)$.
- Finally apply summation operation to get the convoluted sequence.

$$x(n) = x_1(n) * x_2(n) = \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(n-m)$$

	m	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**	
Given Sequences	**$x_1(m)$**					1	2	3	4		
	$x_2(m)$					5	6	7	8	9	
Folding	**$x_2(-m)$**	9	8	7	6	5	0	0	0		
Shifting	**$x_2(1-m)$**		9	8	7	6	5	0	0		
	$x_2(2-m)$			9	8	7	6	5	0		
	$x_2(3-m)$				9	8	7	6	5		
	$x_2(4-m)$					9	8	7	6		
	$x_2(5-m)$					0	9	8	7		
	$x_2(6-m)$					0	0	9	8		
	$x_2(7-m)$					0	0	0	9		**Sum**
Multiplication	**$x_1(m)\, x_2(-m)$**					5	0	0	0	=	**5**
	$x_1(m)\, x_2(1-m)$					6	10	0	0	=	**16**
	$x_1(m)\, x_2(2-m)$					7	12	15	0	=	**34**
	$x_1(m)\, x_2(3-m)$					8	14	18	20	=	**60**
	$x_1(m)\, x_2(4-m)$					9	16	21	24	=	**70**
	$x_1(m)\, x_2(5-m)$					0	18	24	28	=	**70**
	$x_1(m)\, x_2(6-m)$					0	0	27	32	=	**59**
	$x_1(m)\, x_2(7-m)$					0	0	0	36	=	**36**

Convoluted Sequence $x(n) = x_1(n) * x_2(n) = \{5, 16, 34, 60, 70, 70, 59, 36\}$

(c)Matrix method:

- Pad $x_1(m)$ '4' number of zeros to get a length of 8 samples.

  $x_1(n) = \{1,2,3,4,0,0,0,0\}$
- Pad $x_2(m)$ with '3' number of zeros to get a length of 8 samples.

  $x_2(n) =\{5,6,7,8,9,0,0,0\}$
- Now represent $x_1(m)$ and $x_2(n-m)$ in the form of matrices and finally compute the convoluted sequence.

$$x(n) = x_1(n) * x_2(n) = \sum_{m=0}^{7} x_1(m)\, x_2(n-m)$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 6 & 7 & 8 & 9 & 0 & 0 & 0 \\ 0 & 5 & 6 & 7 & 8 & 9 & 0 & 0 \\ 0 & 0 & 5 & 6 & 7 & 8 & 9 & 0 \\ 0 & 0 & 0 & 5 & 6 & 7 & 8 & 9 \\ 9 & 0 & 0 & 0 & 5 & 6 & 7 & 8 \\ 8 & 9 & 0 & 0 & 0 & 5 & 6 & 7 \\ 7 & 8 & 9 & 0 & 0 & 0 & 5 & 6 \\ 6 & 7 & 8 & 9 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$$= [1\times5 \quad 1\times6+2\times5 \quad 1\times7+2\times6+3\times5 \quad 1\times8+2\times7+3\times6+4\times5$$
$$1\times9+2\times8+3\times7+4\times6 \quad 2\times9+3\times8+4\times7 \quad 3\times9+4\times8 \quad 4\times9]$$
$$= [5 \quad 6+10 \quad 7+12+15 \quad 8+14+18+20$$
$$9+16+21+24 \quad 18+24+28 \quad 27+32 \quad 36]$$
$$= [5 \quad 16 \quad 34 \quad 60 \quad 70 \quad 70 \quad 59 \quad 36]$$

Convoluted Sequence $x(n) = x_1(n) * x_2(n) = \{5, 16, 34, 60, 70, 70, 59, 36\}$

Example-2:

Compute the convoluted sequence $x(n) = x_1(n) * x_2(n)$, if (i) b=a. (i) b≠a

Given $x_1(n)=a^n u(n)$ and $x_2(n)=b^n u(n)$.

$$x(n) = x_1(n) * x_2(n)$$
$$= \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(n-m)$$
$$= \sum_{m=-\infty}^{\infty} a^m u(m)\, b^{n-m} u(n-m)$$
$$= \sum_{m=0}^{n} a^m b^{n-m}$$
$$= b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m$$

$u(m) = 1,\ m \ge 0$ and $u(n-m) = 1,\ n - m \ge 0$ or $m \le n$ $u(m)u(n-m) = 1,\ 0 \le m \le n$

Case-1: if b=a

$$x(n) = x_1(n) * x_2(n)$$

$$= \quad a^n \sum_{m=0}^{n} (1)^m$$

$$= a^n (1+1+1+........1, Add, n+1, times)$$

$$= (n+1)a^n, n \geq 0$$

Case-2: if b≠a

$$x(n) = x_1(n) * x_2(n)$$

$$= \quad b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m$$

$$= b^n \frac{1-\left(\frac{a}{b}\right)^{n+1}}{1-\frac{a}{b}}$$

$$= \frac{b^{n+1} - a^{n+1}}{b-a}, n \geq 0$$

Classification or Properties of Discrete Time Signals:

Classification or various properties of discrete time signals are given below

- Symmetric (Even) and Anti-symmetric (Odd) Signals
- Causal and Non-causal Signals
- Bounded and Unbounded Signals
- Periodic and Non-periodic Signals
- Energy and Power Signals
- Deterministic and Nondeterministic Signals

(A)Symmetric (Even) and Anti-symmetric (Odd) Signals:

- Discrete time signal x(n) is said to be even only when x(–n) = x(n). Even signals are symmetrical about y – axis, hence even signals are called symmetric signals.
- Discrete time signal x(n) is said to be odd only when x(–n) = -x(n). Odd signals are anti-symmetrical about y – axis, hence odd signals are called anti-symmetric signals.
- If a discrete time signal fails to satisfy even and odd property, then the signal is neither even nor odd and it can be expressed as a sum of even signal $x_e(n)$ and odd signal $x_o(n)$.

  $x(n) = x_e(n) + x_o(n)$ --------------- (1)

  Replace n with –n

  $x(-n) = x_e(-n) + x_o(-n)$

$x(-n) = x_e(n) - x_o(n)$ --------------- (2)

(1)+(2) $\Rightarrow x(n) + x(-n) = x_e(n) + x_o(n) + x_e(n) - x_o(n)$

$$\Rightarrow x_e(n) = \frac{x(n)+x(-n)}{2}$$

(1)-(2) $\Rightarrow x(n) - x(-n) = x_e(n) + x_o(n) - (x_e(n) - x_o(n))$

$$\Rightarrow x_o(n) = \frac{x(n)-x(-n)}{2}$$

Examples:

- $x(n) = Cos(0.125\pi n)$ is an even signal.
- $y(n) = Sin(0.125\pi n)$ is an odd signal.
- $z(n) = Cos(0.125\pi n) + Sin(0.125\pi n)$ is neither even nor odd signal.
- Determine the even and odd parts of following signals

(1)$x(n)=2^n$ (2) $y(n) = 3e^{j\frac{\pi}{5}n}$ (3) $z(n) = \left\{1,2,3,\underset{\uparrow}{4},5,6,7,8,9\right\}$

Problem-1:

Given discrete time signal $x(n)=2^n$

Even part of the signal $x_e(n) = \frac{x(n)+x(-n)}{2} = \frac{3^n + 3^{-n}}{2}$

Odd part of the signal $x_o(n) = \frac{x(n)-x(-n)}{2} = \frac{3^n - 3^{-n}}{2}$

Problem-2:

Given discrete time signal $y(n) = 3e^{j\frac{\pi}{5}n} = 3Cos\left(\frac{\pi}{5}n\right) + j3Sin\left(\frac{\pi}{5}n\right)$

$$y(-n) = 3e^{-j\frac{\pi}{5}n} = 3Cos\left(\frac{\pi}{5}n\right) - j3Sin\left(\frac{\pi}{5}n\right)$$

Even part of the signal $y_e(n) = \frac{y(n)+y(-n)}{2} = 3Cos\left(\frac{\pi}{5}n\right)$

Odd part of the signal $y_o(n) = \frac{y(n)-y(-n)}{2} = j3Sin\left(\frac{\pi}{5}n\right)$

Problem-3:

Given discrete time signal $z(n) = \left\{1,2,3,\underset{\uparrow}{4},5,6,7,8,9\right\}$

$$z(-n) = \left\{9,8,7,6,5,\underset{\uparrow}{4},3,2,1\right\}$$

Even part of the signal

$$z_e(n) = \frac{z(n)+z(-n)}{2} = \frac{1}{2}\left\{9,8,8,8,8,\underset{\uparrow}{8},8,8,8,8,9\right\} = \left\{4.5,4,4,4,4,\underset{\uparrow}{4},4,4,4,4,4.5\right\}$$

Odd part of the signal

$$z_o(n) = \frac{z(n)-z(-n)}{2} = \frac{1}{2}\left\{-9,-8,-6,-4,-2,\underset{\uparrow}{0},2,4,6,8,9\right\} = \left\{-4.5,-4,-3,-2,-1,\underset{\uparrow}{0},1,2,3,4,4.5\right\}$$

(B)Causal and Non-causal Signals:

- A discrete time signal x(n) is said be **causal** only when x(n) = 0; for n < 0, that means causal signals are right sided.
- A discrete time signal x(n) is said be **anti-causal** only when x(n) = 0; for n > 0, that means anti-causal signals are left sided.
- A discrete time signal x(n) is said be **non-causal** only when x(n) ≠ 0; for n < 0, that means non-causal signals may be left sided or both sided and all anti-causal signals are comes under non-causal signals.

Examples:

- x(n) = u(n) is a causal signal.
- y(n) = u(-n) is anti-causal and non-causal signal.
- z(n) = u(-n-1) is anti-causal and non-causal signal.
- x(n) = u(n+1) is non-causal signal.

(C)Bounded and Unbounded Signals:

A discrete time signal x(n) is said be bounded only when the amplitude of x(n) is finite for all values of n over the range $-\infty \le n \le \infty$. Condition for a bounded signal is $|x(n)| < \infty, -\infty \le n \le \infty$.

Examples:

- Bounded Signals : $\delta(n)$, u(n), $0.5^n u(n)$, $2^n u(-n-1)$, $0.5^{|n|}$
- Unbounded Signals : r(n), $2^n u(n)$, $0.5^n u(-n-1)$, $2^{|n|}$

(D)Periodic and Aperiodic Signals:

A discrete time signal x(n) is said be periodic if and only if x(n+N) = x(n), otherwise the signal is aperiodic or non-periodic, where N is smallest positive integer and it is called fundamental period or period of given signal x(n).

Example-1: $x(n) = 9Cos\left(\frac{2\pi}{9}n + 3\right)$

If x(n) is periodic with a period of N samples, then

$$x(n+N) = 9Cos\left(\frac{2\pi}{9}(n+N)+3\right) = 9Cos\left(\frac{2\pi}{9}n + \frac{2\pi}{9}N + 3\right) = 9Cos\left(\frac{2\pi}{9}n + 3 + 2\pi k\right) = x(n)$$

$$\Rightarrow \frac{2\pi}{9}N = 2\pi k \Rightarrow N = 9k$$

Possible integer values of N = 9,18,27,…. for integer values of k = 1,2,3….

Hence, given signal is periodic with a period N=9 samples.

Example-2: $x(n) = 9Cos\left(\frac{5\pi}{9}n + 3\right)$

If x(n) is periodic with a period of N samples, then

$$x(n+N) = 9Cos\left(\frac{5\pi}{9}(n+N)+3\right) = 9Cos\left(\frac{5\pi}{9}n + \frac{5\pi}{9}N + 3\right) = 9Cos\left(\frac{5\pi}{9}n + 3 + 2\pi k\right) = x(n)$$

$$\Rightarrow \frac{5\pi}{9}N = 2\pi k \Rightarrow N = \frac{18}{5}k$$

Possible integer values of N = 18,36,54,…. for integer values of k = 5,10,15….

Hence, given signal is periodic with a period N=18 samples.

Example-3: $x(n) = Sin\left(\frac{5}{9}n + \pi\right)$

If x(n) is periodic with a period of N samples, then

$$x(n+N) = Sin\left(\frac{5}{9}(n+N)+\pi\right) = Sin\left(\frac{5}{9}n + \frac{5}{9}N + \pi\right) = Sin\left(\frac{5}{9}n + \pi + 2\pi k\right) = x(n)$$

$\Rightarrow \frac{5}{9}N = 2\pi k \Rightarrow N = \frac{18\pi}{5}k$, Integer value of N is not possible for integer values of k.

Hence, given signal is aperiodic or non-periodic.

Example-4: $x(n)=x_1(n)+x_2(n)=3Cos\left(\frac{8\pi}{3}n+4\right)+2Sin\left(\frac{3\pi}{2}n+5\right)$

If $x_1(n)$, $x_2(n)$ and x(n) are periodic with a periods of N_1, N_2 and N samples, then x(n+N)=x(n)

$$\Rightarrow x_1(n+N_1)=3Cos\left(\frac{8\pi}{3}(n+N)+4\right)=3Cos\left(\frac{8\pi}{3}n+\frac{8\pi}{3}N+4\right)=3Cos\left(\frac{8\pi}{3}n+4+2\pi k\right)=x_1(n)$$

$$\Rightarrow \frac{8\pi}{3}N=2\pi k\Rightarrow N=\frac{3}{4}k$$

Possible integer values of N_1 = 3,6,9,12,… for integer values of k = 4,8,12,16…. and

$$\Rightarrow x_2(n+N_2)=2Sin\left(\frac{3\pi}{2}(n+N)+5\right)=2Sin\left(\frac{3\pi}{2}n+\frac{3\pi}{2}N+5\right)=2Sin\left(\frac{3\pi}{2}n+5+2\pi k\right)=x_2(n)$$

$$\Rightarrow \frac{3\pi}{2}N=2\pi k\Rightarrow N=\frac{4}{3}k$$

Possible integer values of N_2 = 4,8,12,16… for integer values of k = 3,6,9,12….

N=Common & Min (N_1,N_2)=12, hence, given signal is periodic with a period N=12 samples.

Example-5: $x(n)=3Sin\left(\frac{\pi}{8}n^2\right)$

If x(n) is periodic with a period of N samples, then

$$x(n+N)=3Sin\left(\frac{\pi}{8}(n+N)^2\right)=3Sin\left(\frac{\pi}{8}n^2+\frac{\pi}{8}N^2+\frac{\pi}{4}Nn\right)=9Cos\left(\frac{\pi}{8}n^2+2\pi k_1+2\pi k_2 n\right)=x(n)$$

$$\Rightarrow \frac{\pi}{8}N^2=2\pi k_1\Rightarrow N^2=16k_1\Rightarrow N=4\sqrt{k_1}$$

Possible integer values of N = 4,8,12,16… for integer values of k_1 = 1,4,9,16…. and

$$\Rightarrow \frac{\pi}{4}N=2\pi k_2\Rightarrow N=8k_2$$

Possible integer values of N = 8,16… for integer values of k_2 = 1,2….

N=Common & Min (N of k_1,N of k_2)=8, hence, given signal is periodic with a period N=8 samples.

Example-6: What is the time period of a signal x(n), which is the sum of 3 periodic signals with periods N_1=2, N_2=3 and N_3=5.

Possible integer values if k = 1,2,3,4,5,6,7,8,9,….

N_1k=2,4,6,8,10,12,14,16,18,20,22,24,26,28,30,32,34……

N_2k=3,6,9,12,15,18,21,24,27,30,33………

N_3k=5,10,15,20,25,30,35,40,45…….

N=Common & Min (N_1,N_2,N_3)=30, hence, given signal is periodic with a period N=30 samples.

(E)Energy and Power Signals:

- A discrete time signal x(n) is said be energy signal only when the total energy (E) under the signal x(n) is finite and the average power (P) is zero.

  $0 < E < \infty$, and $P = 0$

- A discrete time signal x(n) is said be power signal only when the average power (P) of the signal x(n) is finite and the total energy is infinity.

  $0 < P < \infty$, and $E = \infty$

- If a discrete time signal fails to satisfy energy and power signal properties, then the signal is called neither energy nor power.
- In general periodic signals are power signals and aperiodic signals are energy signals.
- Total energy and average power can be computed from the following formulas

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \ \& \ P = \underset{N\to\infty}{Lt}\left(\frac{1}{2N+1}\right)\sum_{n=-N}^{N}|x(n)|^2$$

Example-1: $x(n) = 3\left(\frac{1}{2}\right)^n u(n)$

Total Energy:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty}\left|3\left(\frac{1}{2}\right)^n u(n)\right|^2 = \sum_{n=0}^{\infty} 9\left(\frac{1}{4}\right)^n = 9\left(\frac{1}{1-1/4}\right) = 9\left(\frac{4}{3}\right) = 12\,joules$$

Average Power:

$$P = \underset{N\to\infty}{Lt}\left(\frac{1}{2N+1}\right)\sum_{n=-N}^{N}|x(n)|^2 = \underset{N\to\infty}{Lt}\left(\frac{1}{2N+1}\right)\sum_{n=-N}^{N}\left|3\left(\frac{1}{2}\right)^n u(n)\right|^2 = \underset{N\to\infty}{Lt}\left(\frac{1}{2N+1}\right)\sum_{n=0}^{N} 9\left(\frac{1}{4}\right)^n$$

$$= \underset{N\to\infty}{Lt}\left(\frac{9}{2N+1}\right)\left(\frac{1-\left(1/4\right)^{N+1}}{1-1/4}\right) = \underset{N\to\infty}{Lt}\left(\frac{12}{2N+1}\right)\left(1-\left(1/4\right)^{N+1}\right) = \frac{12}{\infty}(1-0) = \frac{12}{\infty} = 0$$

Given x(n) is energy signal because the total energy is finite and average power is zero.

Example-2: $x(n) = u(n)$

Total Energy: $E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} |u(n)|^2 = \sum_{n=0}^{\infty} 1 = \infty$

Average Power:

$$P = \underset{N\to\infty}{Lt}\left(\frac{1}{2N+1}\right)\sum_{n=-N}^{N}|x(n)|^2 = \underset{N\to\infty}{Lt}\left(\frac{1}{2N+1}\right)\sum_{n=-N}^{N}|u(n)|^2 = \underset{N\to\infty}{Lt}\left(\frac{1}{2N+1}\right)\sum_{n=0}^{N}1 = \underset{N\to\infty}{Lt}\left(\frac{N+1}{2N+1}\right) = \frac{1}{2}Watts$$

Given x(n) is power signal because the average power is finite and total energy is infinity.

Example-3: $x(n) = 2Sin\left(\frac{\pi}{3}n\right)$

Total Energy:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} \left| 2Sin\left(\frac{\pi}{3}n\right)\right|^2 = \sum_{n=-\infty}^{\infty} 4Sin^2\left(\frac{\pi}{3}n\right) = 4\sum_{n=-\infty}^{\infty} \frac{1}{2}\left(1 - Cos\left(\frac{2\pi}{3}n\right)\right) = 2(\infty - 0) = \infty$$

Average Power:

$$P = \underset{N\to\infty}{Lt}\left(\frac{1}{2N+1}\right)\sum_{n=-N}^{N} |x(n)|^2 = \underset{N\to\infty}{Lt}\left(\frac{1}{2N+1}\right)\sum_{n=-N}^{N} \left|2Sin\left(\frac{\pi}{3}n\right)\right|^2 = \underset{N\to\infty}{Lt}\left(\frac{4}{2N+1}\right)\sum_{n=-N}^{N} Sin^2\left(\frac{\pi}{3}n\right)$$

$$= \underset{N\to\infty}{Lt}\left(\frac{4}{2N+1}\right)\sum_{n=-N}^{N} \frac{1}{2}\left(1 - Cos\left(\frac{2\pi}{3}n\right)\right) = \underset{N\to\infty}{Lt}\left(\frac{2}{2N+1}\right)(2N+1-0) = 2Watts$$

Given x(n) is power signal because the average power is finite and total energy is infinity.

Example-4: Determine the energy and power of $x(n) = n(-1)^n, n = 1,2,3$ and $y(n) = \sum_{k=-\infty}^{\infty} x(n+7k)$

Given $x(n) = n(-1)^n, n = 1,2,3 = \left\{\underset{\uparrow}{0}, -1, 2, -3\right\} =$

Total Energy: $E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = |-1|^2 + |2|^2 + |-3|^2 = 1 + 4 + 9 = 14 \, jouls$

Average Power:

$$P_x = \underset{N\to\infty}{Lt}\left(\frac{1}{2N+1}\right)\sum_{n=-N}^{N} |x(n)|^2 = \underset{N\to\infty}{Lt}\left(\frac{1}{2N+1}\right)\left(|-1|^2 + |2|^2 + |-3|^2\right) = \underset{N\to\infty}{Lt}\left(\frac{14}{2N+1}\right) = \frac{14}{\infty} = 0$$

Given x(n) is energy signal because the total energy is finite and average power is zero.

Given $y(n) = \sum_{k=-\infty}^{\infty} x(n+7k) = .. + x(n-7) + x(n) + x(n+7) + ..$ periodic with a period of 7 samples

$$y(n) = \left\{ ....,-1,2,-3,0,0,0,\underset{\uparrow}{0},-1,2,-3,0,0,0,0,-1,2,-3,0,0,0,0,-1,2,-3,....... \right\}$$

Total Energy: $E_y = \sum_{n=-\infty}^{\infty} |y(n)|^2 = ... + |-1|^2 + |2|^2 + |-3|^2 + ... = ...14 + 14 + 14... = \infty$

Average Power:

$$P_y = \underset{N\to\infty}{Lt}\left(\frac{1}{2N+1}\right)\sum_{n=-N}^{N} |y(n)|^2 = \left(\frac{0 + |-1|^2 + |2|^2 + |-3|^2 + 0 + 0 + 0}{7}\right) = \frac{14}{7} = 2$$

Given y(n) is power signal because the average power is finite and total energy is infinity.

(F)Deterministic and Nondeterministic Signals:

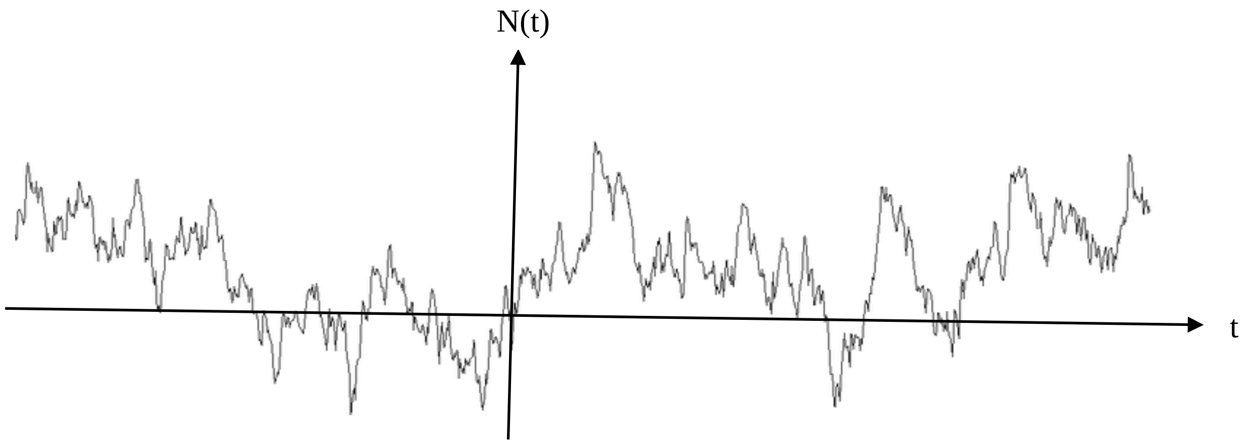
Signals that are completely specified by a mathematical expression are called deterministic signals, where the amplitude of the signal can be determined at any instant of time.

Examples:

- Digital Impulse Signal or Unit Sample Sequence, $\delta(n) = \begin{cases} 1 & ; n = 0 \\ 0 & ; n \neq 0 \end{cases}$
- Unit Step Signal, $u(n) = \begin{cases} 1 & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$
- Unit Ramp Signal, $r(n) = nu(n) = \begin{cases} n & ; n \geq 0 \\ 0 & ; n < 0 \end{cases}$
- Decaying Exponential Signals, $x(n) = a^n u(n), 0 < a < 1$ and $y(n) = a^n, 0 < a < 1$
- Raising Exponential Signals, $x(n) = a^n u(-n-1), a > 1$ and $y(n) = a^n, a > 1$
- Double Exponential Signals, $x(n) = a^{|n|}, 0 < a < 1$ and $y(n) = a^{|n|}, a > 1$

Signals whose characteristics are random in nature are called nondeterministic signals or random signals, where the mathematical representation is not possible, for example noise signal.

Example:



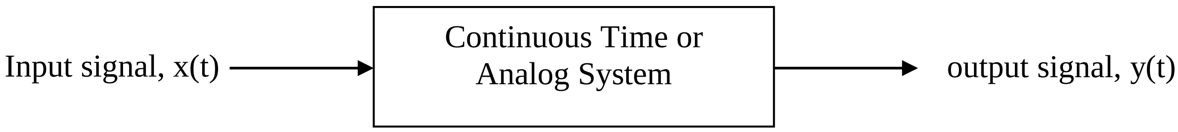
System Definition and Classification:

System can be defined as the collection of objects or elements or components and all these things should be interconnected in such a way to achieve an objective or predefined result or outcome. Based on the type of input applied, components used in the design and type output, systems are classified into two types.

- Continuous Time or Analog Systems
- Discrete Time and Digital Systems

Continuous Time or Analog Systems:

Continuous time or analog systems are those for which both input and output are continuous time signals and are constructed by using analog components, like resistors, capacitors, inductors, diodes, transistors, analog ICs, etc.



- Thermal stability of continuous time or analog systems is poor because of all analog components are temperature sensitive.
- Continuous time or analog systems are non programmable and static in nature.
- Continuous time or analog systems are described by differential equation, which involves only differentials.

Examples:

(a) A simple RC high pass filter acts as a differentiator, where the output y(t) is the differentiation of input x(t)

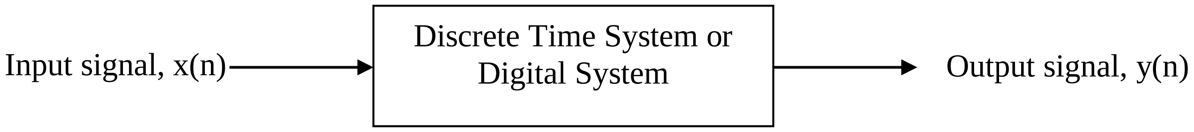
$$y(t) = \frac{d}{dt} [\ x(t)\ ]$$

(b) A simple RC low pass filter acts as an integrator, where the output y(t) is the integration of input x(t)

$$y(t) = \int x(t) \Rightarrow x(t) = \frac{d}{dt}[y(t)]$$

Discrete Time and Digital Systems:

Discrete time systems are those for which both input and output are discrete time signals and are constructed by using discrete components, like adders, constant multipliers and delays (memories). In the case of digital systems, both input and output are digital signals.

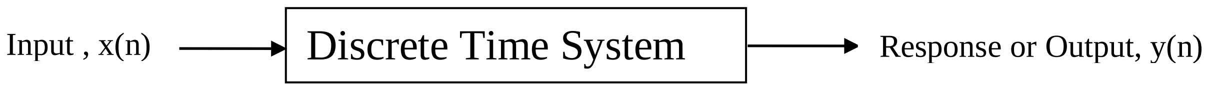


- Thermal stability of discrete time or digital system is high.
- Discrete time or digital systems are programmable and dynamic.
- Discrete time or digital systems are described by a difference equation, which does not involve differentials, which involve only shifts.

Examples:

- $y(n) = 2\,x(n) + 3\,x(n-1) + 4\,x(n-2)$
- $y(n) = 2\,x(n) + 3\,x(n-1) + 4\,y(n-2)$.
- $y(n) = 2\,x(n) + n\,x(n-1) + 4\,y(n-2)$.
- $y(n) = 2\,x(n^2) + 3\,x(n+1) + 4\,y(n-2)$.
- $y(n) = 2\,x(n/2) + 3\,x(n-1) + 4\,y(n-2)$.

Response of Discrete Time System:



(A)Impulse or Unit Sample Response:

Output of discrete time system with an input of impulse or unit sample signal is called impulse response or unit sample response and it is represented with h(n).

Example-1:

Determine the impulse response of a discrete time system y(n) = 2 x(n) + 3 x(n – 1) + 4 x(n – 2).

Put, x(n) = δ(n) and y(n) = h(n)

h(n) = 2 δ(n) + 3 δ(n – 1) + 4δ(n – 2)

n =-1 ⇒ h(-1) = 2 δ(-1) + 3 δ(–2) + 4δ(–3) = 0

n = 0 ⇒ h(0) = 2 δ(0) + 3 δ(–1) + 4δ(–2) = 2

n = 1 ⇒ h(1) = 2 δ(1) + 3 δ(0) + 4δ(–1) = 3

n = 2 ⇒ h(2) = 2 δ(2) + 3 δ(1) + 4δ(0) = 4

n = 3 ⇒ h(3) = 2 δ(3) + 3 δ(2) + 4δ(1) = 0

$$h(n) = \left\{ \underset{\uparrow}{2}, 3, 4 \right\}$$

Example-2: Determine the impulse response of a discrete time system $y(n) - \frac{1}{2} y(n-1) = x(n)$ by assuming zero initial conditions.

Put, x(n) = δ(n) and y(n) = h(n)

$$h(n) = \frac{1}{2} h(n-1) + \delta(n)$$

$$n = -1 \Rightarrow h(-1) = \frac{1}{2} h(-2) + \delta(-1) = \frac{1}{2}(0) + 0 = 0$$

$$n = 0 \Rightarrow h(0) = \frac{1}{2} h(-1) + \delta(0) = \frac{1}{2}(0) + 1 = 1$$

$$n = 1 \Rightarrow h(1) = \frac{1}{2} h(0) + \delta(1) = \frac{1}{2}(1) + 0 = \frac{1}{2}$$

$$n = 2 \Rightarrow h(2) = \frac{1}{2} h(1) + \delta(2) = \frac{1}{2}\left(\frac{1}{2}\right) + 0 = \left(\frac{1}{2}\right)^2$$

$$n = 3 \Rightarrow h(3) = \frac{1}{2} h(2) + \delta(3) = \frac{1}{2}\left(\frac{1}{2}\right)^2 + 0 = \left(\frac{1}{2}\right)^3$$

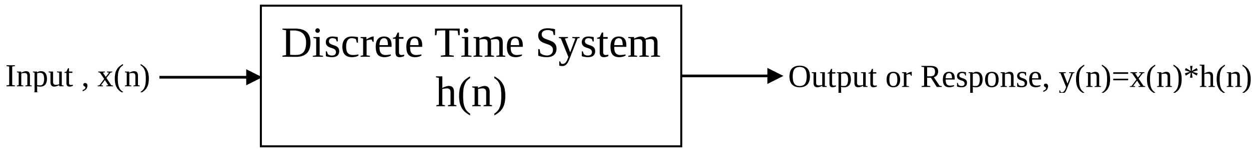
…..

…..

$$h(n) = \left(\frac{1}{2}\right)^n, n \geq 0 \text{ or } h(n) = \left(\frac{1}{2}\right)^n u(n)$$

(B)Response through Convolution:

Response of the discrete time system is the convolution of input x(n) and impulse response h(n).



Example-1: Determine the impulse response and response of the discrete time system $y(n)-ay(n-1)=x(n)$ by assuming zero initial conditions with input $x(n)=b^n u(n)$.

Put, x(n) = δ(n) and y(n) = h(n) in $h(n)=ah(n-1)+\delta(n)$

$n=-1 \Rightarrow h(-1)=ah(-2)+\delta(-1)=a(0)+0=0$

$n=0 \Rightarrow h(0)=ah(-1)+\delta(0)=a(0)+1=1$

$n=1 \Rightarrow h(1)=ah(0)+\delta(1)=a(1)+0=a$

$n=2 \Rightarrow h(2)=ah(1)+\delta(2)=a(a)+0=a^2$

$n=3 \Rightarrow h(3)=ah(2)+\delta(3)=a(a)^2+0=a^3$

…..

…..

$h(n)=a^n, n\geq 0$ or $h(n)=a^n u(n)$,

It is the impulse response of given system and Response of the system can be computed from

$$
\begin{aligned}
y(n) &= x(n)*h(n)\\
&= \sum_{m=-\infty}^{\infty} x(m)\, h(n-m)\\
&= \sum_{m=-\infty}^{\infty} b^m u(m)\, a^{n-m} u(n-m)\\
&= \sum_{m=0}^{n} b^m a^{n-m}\\
&= a^n \sum_{m=0}^{n}\left(\frac{b}{a}\right)^m
\end{aligned}
$$

u(m) = 1, m ≥ 0 and
u(n-m) = 1, n – m ≥ 0 or m ≤ n
u(m)u(n-m) = 1, 0 ≤ m ≤n

Case-1: if b=a

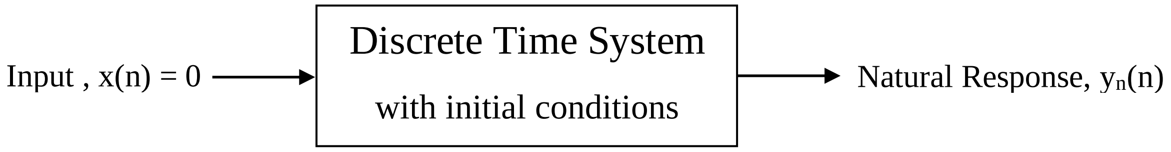
$$y(n)=x(n)*h(n)=a^n\sum_{m=0}^{n}(1)^m = a^n(1+1+1+........1, Add, n+1, times)=(n+1)a^n, n\geq 0, or, (n+1)a^n u(n)$$

Case-2: if b≠a

$$y(n)=x(n)*h(n)= \quad a^n\sum_{m=0}^{n}\left(\frac{b}{a}\right)^m = a^n\frac{1-\left(\frac{b}{a}\right)^{n+1}}{1-\frac{a}{b}} = \frac{a^{n+1}-b^{n+1}}{a-b}, n\geq 0, or, \frac{a^{n+1}-b^{n+1}}{a-b}u(n)$$

(C)Natural and Forced Response:

Response of a discrete time system with zero input and for given initial conditions is called zero input response or free response or natural response.



Step by step process to compute the natural response

Step-1: Obtain the polynomial in r by substituting x(n)=0 and $y(n)=r^n$ in given system.

Step-2: Determine the roots of polynomial. i.e., $r_1,r_2,r_3,\ldots\ldots$

Step-3: Write homogeneous solution $y_h(n)$

(a)If all the roots are different, then $y_h(n) = A(r_1)^n + B(r_2)^n + C(r_3)^n$.

(b)If roots are repeated ($r_1=r_2=r$), then $y_h(n) = (An+B)(r)^n + C(r_3)^n$.

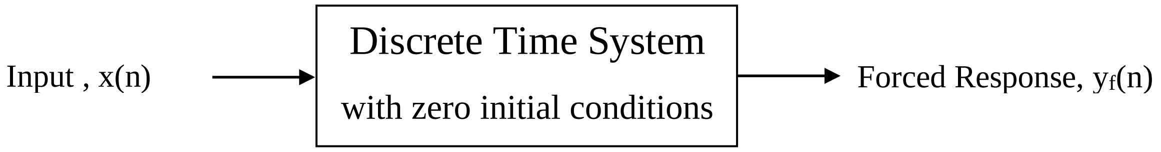
(c)If roots are complex ($r_1=a+jb, r_2=a-jb$), then $y_h(n) = (A\text{Cos}n\theta+B\text{Sin}n\theta)(r)^n + C(r_3)^n$.

Where, $r=\sqrt{a^2+b^2}$ and $\theta=\tan^{-1}\left(\frac{b}{a}\right)$

(c)Find constants A, B, C,… by using initial conditions.

(d)Obtain the natural response by substituting A,B,C,… in Step-3, i.e $y_n(n) = y_h(n)$.

Response of a discrete time system by applying input with zero initial conditions is called zero state response or forced response.



Step by step process to compute the forced response

Step-1: Based on given input, write the particular solution $y_p(n)$.

(a)If $x(n) = \delta(n)$, then $y_p(n) = 0$.

(b)If $x(n) = u(n)$, then $y_p(n) = ku(n)$.

(c)If $x(n) = a^n u(n)$, then $y_p(n) = ka^n u(n)$.

(d)If $x(n) = a^n u(n)$ and any root r_1,r_2,r_3 is equal to ‘a’, then $y_p(n) = kna^n u(n)$.

Step-2: Find the constant ‘k’ by substituting x(n) and $y(n)=y_p(n)$ in given system.

Step-3: Write the forced response $y_f(n)= y_h(n) + y_p(n)$.

Step-4: Find constants A, B, C,… with zero initial conditions.

Step-5: Obtain the forced response by substituting A,B,C,… in Step-3.

Note: Response of a discrete time system is the sum of natural response and forced response.

$$\mathbf{y(n) = y_n(n) + y_f(n).}$$

Example-1: Determine (a)Natural Response (b)Forced Response (c)Response of the system $y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = x(n) + x(n-1)$ with initial conditions y(-1)=y(-2)=1 and input $x(n) = \left(\frac{1}{4}\right)^n u(n)$.

(a)Natural Response: Response of the system with zero input and for given initial conditions

Substitute y(n)=r^n and x(n)=0

$$\Rightarrow r^n - \frac{5}{6}r^{n-1} + \frac{1}{6}r^{n-2} = 0 \Rightarrow r^2 - \frac{5}{6}r + \frac{1}{6} = 0 \Rightarrow \left(r - \frac{1}{2}\right)\left(r - \frac{1}{3}\right) = 0 \Rightarrow r_1 = \frac{1}{2} \,\&\, r_2 = \frac{1}{3}$$

Homogeneous solution of given system

$$\Rightarrow y_h(n) = A(r_1)^n + B(r_2)^n = A\left(\frac{1}{2}\right)^n + B\left(\frac{1}{3}\right)^n$$

Use initial conditions y(-1)=y(-2)=1 to evaluate constants A and B

$$y(n) = y_h(n) = A\left(\frac{1}{2}\right)^n + B\left(\frac{1}{3}\right)^n = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2)$$

$$n = 0 \Rightarrow y(0) = y_h(0) = A\left(\frac{1}{2}\right)^0 + B\left(\frac{1}{3}\right)^0 = \frac{5}{6}y(-1) - \frac{1}{6}y(-2)$$

$$\Rightarrow y(0) = A + B = \frac{5}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3} \Rightarrow 3A + 3B = 2 ------(1)$$

$$n = 1 \Rightarrow y(1) = y_h(1) = A\left(\frac{1}{2}\right)^1 + B\left(\frac{1}{3}\right)^1 = \frac{5}{6}y(0) - \frac{1}{6}y(-1)$$

$$\Rightarrow y(1) = \frac{A}{2} + \frac{B}{3} = \frac{5}{6}\frac{2}{3} - \frac{1}{6} = \frac{10-3}{18} = \frac{7}{18} \Rightarrow 3A + 2B = \frac{7}{3} ----- (2)$$

Solve equations (1) and (2) $\Rightarrow 3B - 2B = 2 - \frac{7}{3} \Rightarrow B = -\frac{1}{3}$ and $A = \frac{2}{3} - B = \frac{2}{3} + \frac{1}{3} = 1$

Natural Response, $y_n(n) = \left(\frac{1}{2}\right)^n - \frac{1}{3}\left(\frac{1}{3}\right)^n, n \geq -2$

(b)Forced Response: Response of the system by applying input with zero initial conditions

Particular solution of given system, $y_p(n) = k\left(\frac{1}{4}\right)^n u(n)$.

Find the constant 'k' by substituting $x(n) = \left(\frac{1}{4}\right)^n u(n) \,\&\, y(n) = y_p(n) = k\left(\frac{1}{4}\right)^n u(n)$ in given system

$$k\left(\frac{1}{4}\right)^{n}u(n)-\frac{5}{6}k\left(\frac{1}{4}\right)^{n-1}u(n-1)+\frac{1}{6}k\left(\frac{1}{4}\right)^{n-2}u(n-2)=\left(\frac{1}{4}\right)^{n}u(n)+\left(\frac{1}{4}\right)^{n-1}u(n-1)$$

$$n=2\Rightarrow k\left(\frac{1}{4}\right)^{2}u(2)-\frac{5}{6}k\left(\frac{1}{4}\right)^{1}u(1)+\frac{1}{6}k\left(\frac{1}{4}\right)^{0}u(0)=\left(\frac{1}{4}\right)^{2}u(2)+\left(\frac{1}{4}\right)^{1}u(1)$$

$$\Rightarrow k\left(\frac{1}{16}\right)-\frac{5}{6}k\left(\frac{1}{4}\right)+\frac{1}{6}k=\left(\frac{1}{16}\right)+\left(\frac{1}{4}\right)\Rightarrow k\left(\frac{1}{16}-\frac{5}{24}+\frac{1}{6}\right)=\frac{5}{16}\Rightarrow k\left(\frac{16}{16}-\frac{5\times 16}{24}+\frac{16}{6}\right)=5$$

$$\Rightarrow k\left(1-\frac{10}{3}+\frac{8}{3}\right)=5\Rightarrow k\left(\frac{3-10+8}{3}\right)=5\Rightarrow k\left(\frac{1}{3}\right)=5\Rightarrow k=15$$

Forced response of given system, $y_f(n)=y_h(n)+y_p(n)=A\left(\frac{1}{2}\right)^{n}+B\left(\frac{1}{3}\right)^{n}+15\left(\frac{1}{4}\right)^{n},n\geq 0$

Find constants A & B with zero initial conditions

$$\Rightarrow y(n)=A\left(\frac{1}{2}\right)^{n}+B\left(\frac{1}{3}\right)^{n}+15\left(\frac{1}{4}\right)^{n}=\frac{5}{6}y(n-1)-\frac{1}{6}y(n-2)+\left(\frac{1}{4}\right)^{n}u(n)+\left(\frac{1}{4}\right)^{n-1}u(n-1)$$

$$n=0\Rightarrow y(0)=A\left(\frac{1}{2}\right)^{0}+B\left(\frac{1}{3}\right)^{0}+15\left(\frac{1}{4}\right)^{0}=\frac{5}{6}y(-1)-\frac{1}{6}y(-2)+\left(\frac{1}{4}\right)^{0}u(0)+\left(\frac{1}{4}\right)^{-1}u(-1)$$

$$\Rightarrow y(0)=A+B+15=\frac{5}{6}(0)-\frac{1}{6}(0)+1+0=1\Rightarrow A+B=-14--------(1)$$

$$n=1\Rightarrow y(1)=A\left(\frac{1}{2}\right)^{1}+B\left(\frac{1}{3}\right)^{1}+15\left(\frac{1}{4}\right)^{1}=\frac{5}{6}y(0)-\frac{1}{6}y(-1)+\left(\frac{1}{4}\right)^{1}u(1)+\left(\frac{1}{4}\right)^{0}u(0)$$

$$\Rightarrow y(1)=\frac{A}{2}+\frac{B}{3}+\frac{15}{4}=\frac{5}{6}(1)-\frac{1}{6}(0)+\frac{1}{4}+1\Rightarrow \frac{A}{2}+\frac{B}{3}=-\frac{15}{4}+\frac{5}{6}+\frac{1}{4}+1$$

$$\Rightarrow 3A+2B=-\frac{15\times 6}{4}+\frac{5\times 6}{6}+\frac{1\times 6}{4}+6=-\frac{45}{2}+5+\frac{3}{2}+6=11+\frac{3-45}{2}=11-21=-10---(2)$$

Solve equations (1) and (2)⇒3A+2(-14-A)=-10⇒3A-2A=28-10⇒A=18 & B=-14-A=-14-18=-32

Forced Response, $y_f(n)=18\left(\frac{1}{2}\right)^{n}-32\left(\frac{1}{3}\right)^{n}+15\left(\frac{1}{4}\right)^{n},n\geq 0$

(c)Response of the System: Sum of natural and forced response

$$y(n)=y_n(n)+y_f(n)=\left(\frac{1}{2}\right)^{n}-\frac{1}{3}\left(\frac{1}{3}\right)^{n}+18\left(\frac{1}{2}\right)^{n}-32\left(\frac{1}{3}\right)^{n}+15\left(\frac{1}{4}\right)^{n},n\geq 0$$

$$\Rightarrow y(n)=19\left(\frac{1}{2}\right)^{n}-\left(\frac{1}{3}+32\right)\left(\frac{1}{3}\right)^{n}+15\left(\frac{1}{4}\right)^{n},n\geq 0$$

$$\Rightarrow y(n)=19\left(\frac{1}{2}\right)^{n}-\frac{97}{3}\left(\frac{1}{3}\right)^{n}+15\left(\frac{1}{4}\right)^{n},n\geq 0$$

Example-2: Determine (a)Natural Response (b)Forced Response (c)Response of the system $y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = x(n) + x(n-1)$ with initial conditions y(-1)=y(-2)=1 and input $x(n) = \left(\frac{1}{2}\right)^n u(n)$.

(a)Natural Response: Response of the system with zero input and for given initial conditions

Substitute y(n)=r^n and x(n)=0

$$\Rightarrow r^n - \frac{5}{6}r^{n-1} + \frac{1}{6}r^{n-2} = 0 \Rightarrow r^2 - \frac{5}{6}r + \frac{1}{6} = 0 \Rightarrow \left(r - \frac{1}{2}\right)\left(r - \frac{1}{3}\right) = 0 \Rightarrow r_1 = \frac{1}{2} \;\&\; r_2 = \frac{1}{3}$$

Homogeneous solution of given system

$$\Rightarrow y_h(n) = A(r_1)^n + B(r_2)^n = A\left(\frac{1}{2}\right)^n + B\left(\frac{1}{3}\right)^n$$

Use initial conditions y(-1)=y(-2)=1 to evaluate constants A and B

$$y(n) = y_h(n) = A\left(\frac{1}{2}\right)^n + B\left(\frac{1}{3}\right)^n = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2)$$

$$n = 0 \Rightarrow y(0) = y_h(0) = A\left(\frac{1}{2}\right)^0 + B\left(\frac{1}{3}\right)^0 = \frac{5}{6}y(-1) - \frac{1}{6}y(-2)$$

$$\Rightarrow y(0) = A + B = \frac{5}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3} \Rightarrow 3A + 3B = 2 ------(1)$$

$$n = 1 \Rightarrow y(1) = y_h(1) = A\left(\frac{1}{2}\right)^1 + B\left(\frac{1}{3}\right)^1 = \frac{5}{6}y(0) - \frac{1}{6}y(-1)$$

$$\Rightarrow y(1) = \frac{A}{2} + \frac{B}{3} = \frac{5}{6}\frac{2}{3} - \frac{1}{6} = \frac{10-3}{18} = \frac{7}{18} \Rightarrow 3A + 2B = \frac{7}{3} ----- (2)$$

Solve equations (1) and (2) $\Rightarrow 3B - 2B = 2 - \frac{7}{3} \Rightarrow B = -\frac{1}{3}$ and $A = \frac{2}{3} - B = \frac{2}{3} + \frac{1}{3} = 1$

Natural Response, $y_n(n) = \left(\frac{1}{2}\right)^n - \frac{1}{3}\left(\frac{1}{3}\right)^n, n \geq -2$

(b)Forced Response: Response of the system by applying input with zero initial conditions

Particular solution of given system, $y_p(n) = kn\left(\frac{1}{2}\right)^n u(n)$.

Find the constant 'k' by substituting $x(n) = \left(\frac{1}{2}\right)^n u(n) \;\&\; y(n) = y_p(n) = kn\left(\frac{1}{2}\right)^n u(n)$ in given system

$$kn\left(\frac{1}{2}\right)^{n}u(n)-\frac{5}{6}k(n-1)\left(\frac{1}{2}\right)^{n-1}u(n-1)+\frac{1}{6}k(n-2)\left(\frac{1}{2}\right)^{n-2}u(n-2)=\left(\frac{1}{2}\right)^{n}u(n)+\left(\frac{1}{2}\right)^{n-1}u(n-1)$$

$$n=2\Rightarrow k(2)\left(\frac{1}{2}\right)^{2}u(2)-\frac{5}{6}k(1)\left(\frac{1}{2}\right)^{1}u(1)+\frac{1}{6}k(0)\left(\frac{1}{2}\right)^{0}u(0)=\left(\frac{1}{2}\right)^{2}u(2)+\left(\frac{1}{2}\right)^{1}u(1)$$

$$\Rightarrow k\left(\frac{1}{2}\right)-\frac{5}{6}k\left(\frac{1}{2}\right)+0=\left(\frac{1}{4}\right)+\left(\frac{1}{2}\right)\Rightarrow k\left(\frac{1}{2}-\frac{5}{12}\right)=\frac{3}{4}\Rightarrow k\left(\frac{6-5}{12}\right)=\frac{3}{4}\Rightarrow k=9$$

Forced response of given system, $y_f(n)=y_h(n)+y_p(n)=A\left(\frac{1}{2}\right)^{n}+B\left(\frac{1}{3}\right)^{n}+9n\left(\frac{1}{2}\right)^{n}, n\geq 0$

Find constants A & B with zero initial conditions

$$\Rightarrow y(n)=A\left(\frac{1}{2}\right)^{n}+B\left(\frac{1}{3}\right)^{n}+9n\left(\frac{1}{2}\right)^{n}=\frac{5}{6}y(n-1)-\frac{1}{6}y(n-2)+\left(\frac{1}{2}\right)^{n}u(n)+\left(\frac{1}{2}\right)^{n-1}u(n-1)$$

$$n=0\Rightarrow y(0)=A\left(\frac{1}{2}\right)^{0}+B\left(\frac{1}{3}\right)^{0}+9(0)\left(\frac{1}{2}\right)^{0}=\frac{5}{6}y(-1)-\frac{1}{6}y(-2)+\left(\frac{1}{2}\right)^{0}u(0)+\left(\frac{1}{2}\right)^{-1}u(-1)$$

$$\Rightarrow y(0)=A+B+0=\frac{5}{6}(0)-\frac{1}{6}(0)+1+0=1\Rightarrow A+B=1-------(1)$$

$$n=1\Rightarrow y(1)=A\left(\frac{1}{2}\right)^{1}+B\left(\frac{1}{3}\right)^{1}+9(1)\left(\frac{1}{2}\right)^{1}=\frac{5}{6}y(0)-\frac{1}{6}y(-1)+\left(\frac{1}{2}\right)^{1}u(1)+\left(\frac{1}{2}\right)^{0}u(0)$$

$$\Rightarrow y(1)=\frac{A}{2}+\frac{B}{3}+\frac{9}{2}=\frac{5}{6}(1)-\frac{1}{6}(0)+\frac{1}{2}+1\Rightarrow\frac{A}{2}+\frac{B}{3}=-\frac{9}{2}+\frac{5}{6}+\frac{1}{2}+1$$

$$\Rightarrow 3A+2B=-\frac{9\times 6}{2}+\frac{5\times 6}{6}+\frac{1\times 6}{2}+6=-27+5+3+6=-13-----(2)$$

Solve equations (1) and (2)

$\Rightarrow$3A+2(1-A)=-13$\Rightarrow$3A-2A=-2-13$\Rightarrow$A=-15 & B=1-A=1+15=16

Forced Response, $y_f(n)=-15\left(\frac{1}{2}\right)^{n}+16\left(\frac{1}{3}\right)^{n}+9n\left(\frac{1}{2}\right)^{n}=(9n-15)\left(\frac{1}{2}\right)^{n}+16\left(\frac{1}{3}\right)^{n}, n\geq 0$

(c)Response of the System: Sum of natural and forced response

$$y(n)=y_n(n)+y_f(n)=\left(\frac{1}{2}\right)^{n}-\frac{1}{3}\left(\frac{1}{3}\right)^{n}+(9n-15)\left(\frac{1}{2}\right)^{n}+16\left(\frac{1}{3}\right)^{n}, n\geq 0$$

$$\Rightarrow y(n)=(9n-14)\left(\frac{1}{2}\right)^{n}+\left(16-\frac{1}{3}\right)\left(\frac{1}{3}\right)^{n}, n\geq 0$$

Example-3:

Determine the Natural Response of the system $y(n) - \frac{2}{3}y(n-1) + \frac{1}{9}y(n-2) = x(n) + x(n-1)$ with initial conditions y(-1)=y(-2)=1.

Natural Response: Response of the system with zero input and for given initial conditions

Substitute y(n)=r^n and x(n)=0

$$\Rightarrow r^n - \frac{2}{3}r^{n-1} + \frac{1}{9}r^{n-2} = 0$$

$$\Rightarrow r^2 - \frac{2}{3}r + \frac{1}{9} = 0$$

$$\Rightarrow \left(r - \frac{1}{3}\right)^2 = 0$$

$$\Rightarrow r = r_1 = r_2 = \frac{1}{3}$$

Homogeneous solution of given system

$$\Rightarrow y_h(n) = (A + nB)(r)^n = (A + nB)\left(\frac{1}{3}\right)^n$$

Use initial conditions y(-1)=y(-2)=1 to evaluate constants A and B

$$y(n) = y_h(n) = (A + nB)\left(\frac{1}{3}\right)^n = \frac{2}{3}y(n-1) - \frac{1}{9}y(n-2)$$

$$n = 0 \Rightarrow y(0) = y_h(0) = (A + (0)B)\left(\frac{1}{3}\right)^0 = \frac{2}{3}y(-1) - \frac{1}{9}y(-2)$$

$$\Rightarrow y(0) = A = \frac{2}{3} - \frac{1}{9} = \frac{6-1}{9} = \frac{5}{9} ------(1)$$

$$n = 1 \Rightarrow y(1) = y_h(1) = (A + B)\left(\frac{1}{3}\right)^1 = \frac{2}{3}y(0) - \frac{1}{9}y(-1)$$

$$\Rightarrow y(1) = \frac{A+B}{3} = \frac{2}{3}\left(\frac{5}{9}\right) - \frac{1}{9} = \frac{10-3}{27} = \frac{7}{27} \Rightarrow A + B = \frac{7}{9} -----(2)$$

Solve equations (1) and (2) $\Rightarrow B = \frac{7}{9} - \frac{5}{9} = \frac{2}{9}$

Natural Response, $y_n(n) = \left(\frac{5}{9} + \frac{2}{9}n\right)\left(\frac{1}{3}\right)^n = \frac{1}{9}(5 + 2n)\left(\frac{1}{3}\right)^n = (5 + 2n)\left(\frac{1}{3}\right)^{n+2}, n \ge -2$

Example-4:

Determine the Natural Response of the system $y(n) - 2y(n-1) + 4y(n-2) = x(n) + x(n-1)$ with initial conditions y(-1)=y(-2)=1.

Natural Response: Response of the system with zero input and for given initial conditions

Substitute y(n)=r^n and x(n)=0

$$\Rightarrow r^n - 2r^{n-1} + 4r^{n-2} = 0 \Rightarrow r^2 - 2r + 4 = 0$$

$$\Rightarrow r = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm j2\sqrt{3}}{2} = 1 \pm j\sqrt{3}$$

$$\Rightarrow r_1 = 1 + j\sqrt{3} \ \& \ r_2 = 1 - j\sqrt{3}$$

Where
$$r = \sqrt{1+3} = \sqrt{4} = 2$$
$$\theta = Tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

Homogeneous solution of given system

$$y_h(n) = (ACosn\theta + BSinn\theta)(r)^n$$

$$\Rightarrow y_h(n) = \left(ACos\frac{n\pi}{3} + BSin\frac{n\pi}{3}\right)(2)^n$$

Use initial conditions y(-1)=y(-2)=1 to evaluate constants A and B

$$y(n) = y_h(n) = \left(ACos\frac{n\pi}{3} + BSin\frac{n\pi}{3}\right)(2)^n = 2y(n-1) - 4y(n-2)$$

$$n = 0 \Rightarrow y(0) = (ACos0 + BSin0)(2)^0 = 2y(-1) - 4y(-2)$$

$$\Rightarrow y(0) = A = 2 - 4 = -2$$

$$n = 1 \Rightarrow y(1) = \left(ACos\frac{\pi}{3} + BSin\frac{\pi}{3}\right)(2)^1 = 2y(0) - 4y(-1)$$

$$\Rightarrow \left(A\left(\frac{1}{2}\right) + B\left(\frac{\sqrt{3}}{2}\right)\right)2 = 2(-2) - 4$$

$$\Rightarrow \left(-2 + B\sqrt{3}\right) = -4 - 4 = -8$$

$$\Rightarrow B\sqrt{3} = 2 - 8 = -6$$

$$\Rightarrow B = -\frac{6}{\sqrt{3}} = -2\sqrt{3}$$

Natural Response,

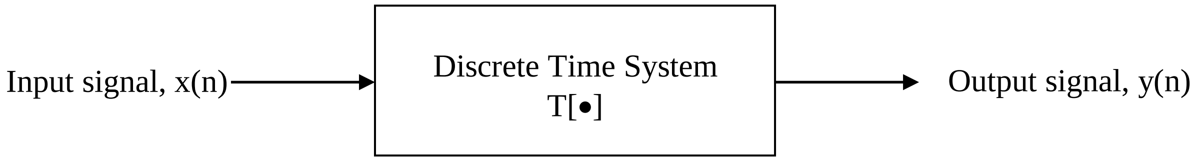
$$y_n(n) = \left(-2Cos\frac{n\pi}{3} - 2\sqrt{3}Sin\frac{n\pi}{3}\right)(2)^n; n \geq -2$$

$$y_n(n) = -2\left(Cos\frac{n\pi}{3} + \sqrt{3}Sin\frac{n\pi}{3}\right)(2)^n; n \geq -2$$

$$y_n(n) = -\left(Cos\frac{n\pi}{3} + \sqrt{3}Sin\frac{n\pi}{3}\right)(2)^{n+1}; n \geq -2$$

Classification or Properties of Discrete Time Systems:

General representation of discrete time system is given below



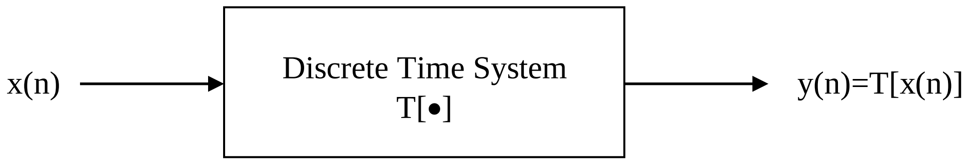
Where T[•] is transform operator and the relation between input x(n) and output y(n) of a discrete time system is represented with $y(n) = T[\,x(n)]$, it shows the output y(n) is the transformation of input x(n).

Various properties or classification of discrete time systems are given below

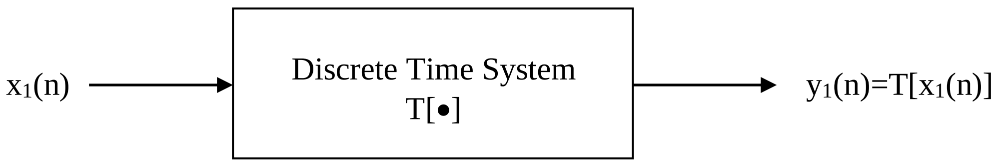
- Linear and Nonlinear Systems
- Shift Invariant and Variant Systems
- Static and Dynamic Systems
- Causal and Noncausal Systems
- Stable and Unstable Systems

(A)Linear and Non Linear Systems:

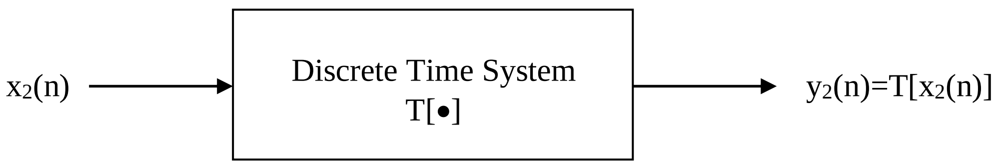
General representation of discrete time system is given below



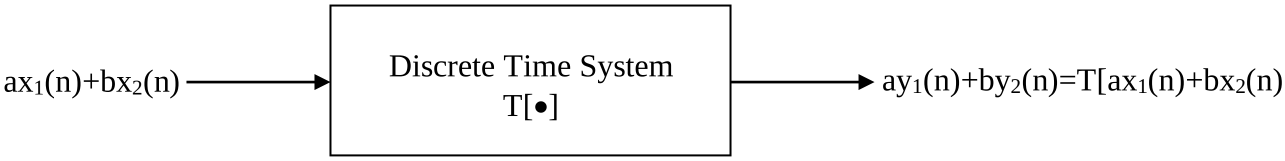
Apply $x_1(n)$ as input to the system T[•] and observe the output, take it as $y_1(n)$



Apply $x_2(n)$ as input to the system T[•] and observe the output, take it as $y_2(n)$



Now apply the linear combination of previous inputs $x_1(n)$ and $x_2(n)$, i.e. $ax_1(n)+bx_2(n)$ as input to the system T[•] and observe the output. If the output is $ay_1(n)+by_2(n)$, then the given system is linear otherwise the system is nonlinear.



- Condition for a linear system is $ay_1(n)+by_2(n)=T[ax_1(n)+bx_2(n)]$.
- Every linear system must satisfy the superposition principle.
- Linearity is the combination of Additivity and Homogenity.
- Additivity $\Rightarrow T[x_1(n)+x_2(n)]= T[x_1(n)]+T[x_2(n)]=y_1(n)+y_2(n)$.
- Homogenity $\Rightarrow T[kx(n)]=k\,T[x(n)]=ky(n)$.

Example-1: Test the discrete time system $y(n) = T[\,x(n)] = 2x(n)+3$ for linearity.

Given discrete time system $y(n) = T[\,x(n)] = 2x(n)+3$

Apply $x_1(n)$ as input to the system $y(n) = T[\,x(n)] = 2x(n)+3$ and observe the output, take it as $y_1(n)$

$y_1(n) = T[\,x_1(n)] = 2x_1(n)+3$

Apply $x_2(n)$ as input to the system $y(n) = T[\,x(n)] = 2x(n)+3$ and observe the output, take it as $y_2(n)$

$y_2(n) = T[\,x_2(n)] = 2x_2(n)+3$

$ay_1(n)+by_2(n)=a(2x_1(n)+3)+b(2x_2(n)+3)= 2ax_1(n)+3a+2bx_2(n)+3b=2ax_1(n)+2bx_2(n)+3a+3b$---(1)

Apply $ax_1(n)+bx_2(n)$ as input to the system $y(n) = T[\,x(n)] = 2x(n)+3$ and observe the output

$T[ax_1(n)+bx_2(n)]=2(ax_1(n)+bx_2(n))+3=2ax_1(n)+2bx_2(n)+3$---(2)

Compare equations (1) and (2) $\Rightarrow ay_1(n)+by_2(n)\neq T[ax_1(n)+bx_2(n)] \Rightarrow$ Given system is nonlinear.

Example-2: Test the discrete time system $y(n) = T[\,x(n)] = 2x(n)+3x(n-1)$ for linearity.

Given discrete time system $y(n) = T[\,x(n)] = 2x(n)+3x(n-1)$

Apply $x_1(n)$ as input to the system $y(n) = T[\,x(n)] = 2x(n)+ 3x(n-1)$ and observe the output, take it as $y_1(n) \Rightarrow y_1(n) = T[\,x_1(n)] = 2x_1(n)+3x_1(n-1)$

Apply $x_2(n)$ as input to the system $y(n) = T[\,x(n)] = 2x(n)+3x(n-1)$ and observe the output, take it as $y_2(n) \Rightarrow y_2(n) = T[\,x_2(n)] = 2x_2(n)+3x_2(n-1)$

$ay_1(n)+by_2(n)=a(2x_1(n)+3x_1(n-1))+b(2x_2(n)+ 3x_2(n-1))= 2ax_1(n)+3ax_1(n-1)+2bx_2(n)+3bx_2(n-1)$

$\Rightarrow ay_1(n)+by_2(n) =2ax_1(n)+2bx_2(n)+3ax_1(n-1)+3bx_2(n-1)$---(1)

Apply $ax_1(n)+bx_2(n)$ as input to the system $y(n) = T[\,x(n)] = 2x(n)+3x(n-1)$ and observe the output

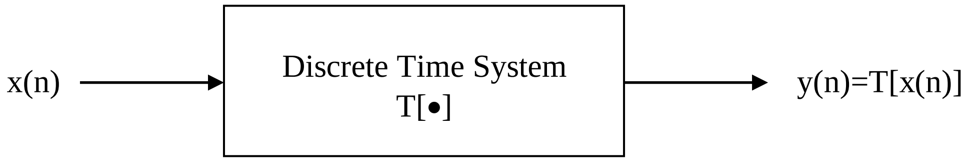
$T[ax_1(n)+bx_2(n)]=2(ax_1(n)+bx_2(n))+3(ax_1(n-1)+bx_2(n-1))$

$\Rightarrow T[ax_1(n)+bx_2(n)]=2ax_1(n)+2bx_2(n)+3ax_1(n-1)+3b\,x_2(n-1)$---(2)

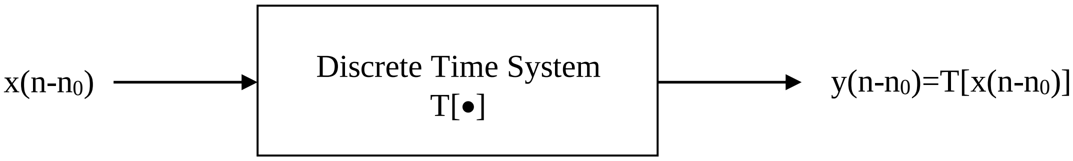
Compare equations (1) and (2) $\Rightarrow ay_1(n)+by_2(n)=T[ax_1(n)+bx_2(n)] \Rightarrow$ Given system is linear.

(B)Shift Invariant and Variant Systems:

General representation of discrete time system is given below



Apply delayed form x(n), i.e $x(n-n_0)$ as input to the system T[•] and observe the output, if the output is $y(n-n_0)$, then the system is shift invariant otherwise the system is shift variant.



Condition for a shift invariant system is $y(n-n_0)=T[x(n-n_0)]$ and it is also known as time invariant.

Example-1: Test the discrete time system y(n) = T[x(n)] = 2x(n)+3 for time invariance.

Given discrete time system T[x(n)] = 2x(n)+3

Apply delayed form x(n), i.e $x(n-n_0)$ as input to the system T[x(n)] = 2x(n)+3 and observe the output $\Rightarrow T[x(n-n_0)] = 2x(n-n_0)+3$-------(1)

Output of given discrete time system y(n) = 2x(n)+3

Replace the discrete time 'n' with '$n-n_0$' in the system y(n) = 2x(n)+3 and observe the output

$\Rightarrow y(n-n_0) = 2x(n-n_0)+3$-------(2)

Compare equations (1) and (2) $\Rightarrow y(n-n_0)=T[x(n-n_0)] \Rightarrow$ Given system is time invariant.

Example-2: Test the discrete time system y(n) = T[x(n)] = nx(n)+3 for time invariance.

Given discrete time system T[x(n)] = nx(n)+3

Apply delayed form x(n), i.e $x(n-n_0)$ as input to the system T[x(n)] = nx(n)+3 and observe the output $\Rightarrow T[x(n-n_0)] = nx(n-n_0)+3$-------(1)

Output of given discrete time system y(n) = nx(n)+3

Replace the discrete time 'n' with '$n-n_0$' in the system y(n) = 2x(n)+3 and observe the output

$\Rightarrow y(n-n_0) = (n-n_0)x(n-n_0)+3$-------(2)

Compare equations (1) and (2) $\Rightarrow y(n-n_0) \neq T[x(n-n_0)] \Rightarrow$ Given system is time variant.

Example-3: Test the following systems for linear shift invariance

Linear Shift Invariant (LSI) System:

- $y(n) = T[x(n)] = 2x(n) - 3y(n-1)$
- $y(n) = T[x(n)] = 2x(n) - 3y(n+1)$
- $y(n) = T[x(n)] = -\sum_{k=1}^{N} a_K\, y(n-k) + \sum_{k=0}^{M} b_K\, x(n-k)$

Linear Shift Variant (LSV) System:

- $y(n) = T[x(n)] = nx(n)$
- $y(n) = T[x(n)] = n^2x(n)$
- $y(n) = T[x(n)] = 2x(n^2)$
- $y(n) = T[x(n)] = 2x(2n)$
- $y(n) = T[x(n)] = nx(n/2)$
- $y(n) = T[x(n)] = 2x(3\sqrt{n})$

Nonlinear Shift Invariant (NLSI) System:

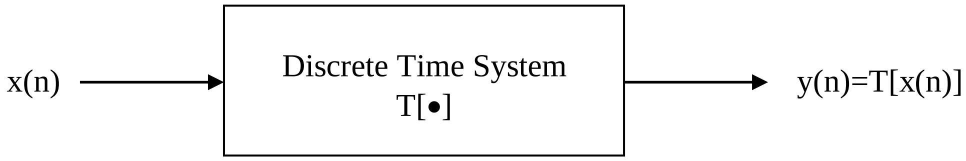
- $y(n) = T[x(n)] = 3x^2(n)$
- $y(n) = T[x(n)] = \text{Constant}$
- $y(n) = T[x(n)] = 2^{x(n)}$
- $y(n) = T[x(n)] = e^{x(n)}$
- $y(n) = T[x(n)] = A\cos[x(n)]$
- $y(n) = T[x(n)] = \log|x(n)|$
- $y(n) = T[x(n)] = \dfrac{2}{x(n)} + 3x(n-1)$

Nonlinear Shift Variant (NLSV) System:

- $y(n) = T[x(n)] = nx^2(n)$
- $y(n) = T[x(n)] = 2x^2(n^2)$
- $y(n) = T[x(n)] = 2 + nx(n)$
- $y(n) = T[x(n)] = 2^{x(3n)}$
- $y(n) = T[x(n)] = e^{x(n/2)}$
- $y(n) = T[x(n)] = A\cos[x(2n-3)]$
- $y(n) = T[x(n)] = n\log|x(n)|$
- $y(n) = T[x(n)] = \dfrac{2}{x(2n)}$
- $y(n) = T[x(n)] = \dfrac{n}{x(n)}$

(C)Static and Dynamic Systems:

General representation of discrete time system is given below



Static systems are those for which the present output y(n) depends on only present input x(n). Static systems are also known as memory less systems.

Examples:

- $y(n) = T[\,x(n)] = nx(n)$
- $y(n) = T[\,x(n)] = 2x(n)+3x^2(n)$
- $y(n) = T[\,x(n)] = n^2x(n)$
- $y(n) = T[x(n)] = \frac{n}{x(n)} + 2x(n)$
- $y(n) = T[\,x(n)] = 2^{x(n)}$

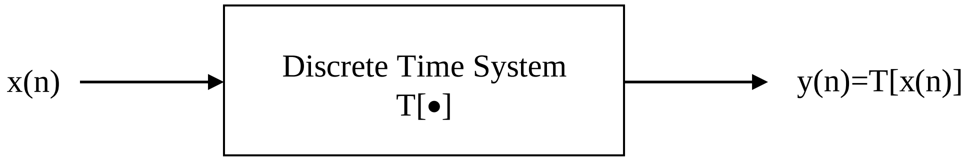
Dynamic systems are those for which the present output y(n) depends on past inputs x(n-1), x(n-2),. and/or past outputs y(n-1), y(n-2),… and/or future inputs x(n+1), x(n+2),…. and/or future outputs y(n+1), y(n+2),….. Dynamic systems are also known as memory systems.

Examples:

- $y(n) = T[\,x(n)] = 2x(n)+3x(n-1)$
- $y(n) = T[\,x(n)] = 2x(n)+3x^2(n-2)$
- $y(n) = T[\,x(n)] = n^2x(n+1)$
- $y(n) = T[x(n)] = \frac{n}{x(n)} + 2y(n-3)$
- $y(n) = T[\,x(n)] = 2^{x(n-5)}$
- $y(n) = T[x(n)] = -\sum_{k=1}^{N} a_K\, y(n-k) + \sum_{k=0}^{M} b_K\, x(n-k)$
- $y(n) = T[x(n)] = \sum_{k=0}^{\infty} b_K\, x(n-k)$

(D)Causal and Noncausal Systems:

General representation of discrete time system is given below



Causal systems are those for which the present output y(n) depends on only present input x(n) and/or past inputs x(n-1), x(n-2),…. and/or past outputs y(n-1), y(n-2),… but does not depends on future inputs x(n+1), x(n+2),…. and/or future outputs y(n+1), y(n+2),….. All static systems are causal.

Non-causal systems are those for which the present output y(n) depends on future inputs x(n+1), x(n+2),…. and/or future outputs y(n+1), y(n+2).

Examples:

- $y(n) = T[\, x(n)] = 2x(n)+3x^2(n)$
- $y(n) = T[\, x(n)] = n^2x(n)+3$
- $y(n) = T[\, x(n)] = 2^{x(n)}$

} Causal and Static (memory less)

- $y(n) = T[\, x(n)] = 2x(n)-3y(n-1)$
- $y(n) = T[\, x(n)] = n^2x(n-6)$
- $y(n) = T[\, x(n)] = 3+2^{x(n-2)}$
- $y(n) = T[x(n)] = -\sum_{k=1}^{N} a_K\, y(n-k) + \sum_{k=0}^{M} b_K\, x(n-k)$

} Causal and Dynamic (memory)

- $y(n) = T[\, x(n)] = 2x(n)+3x^2(n+2)$
- $y(n) = T[\, x(n)] = 2x(n/2)$
- $y(n) = T[\, x(n)] = 2^{x(-n)}$
- $y(n) = T[x(n)] = -\sum_{k=1}^{N} a_K\, y(n+k) + \sum_{k=0}^{M} b_K\, x(n-k)$
- $y(n) = T[x(n)] = \sum_{k=-4}^{\infty} b_K\, x(n-k)$

} Noncausal and Dynamic (memory)

Note: Impulse response h(n) of a causal system is right sided.

Examples:

- $h(n)=(1/2)^n\, u(n)$
- $h(n)=2^n\, u(n)$

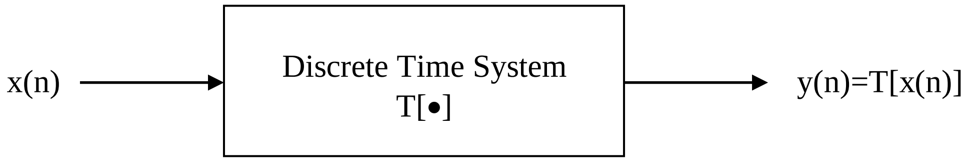
} Causal Systems

- $h(n)=2^n\, u(-n)$
- $h(n)=(1/2)^n\, u(-n)$

} Noncausal Systems

(E)Stable and Unstable Systems:

General representation of discrete time system is given below



Apply bounded signal x(n) as input to the system T[•] and observe the output y(n), if the output is bounded, then the system is stable otherwise the system is unstable. Stable systems are called Bounded Input Bounded Output (BIBO) systems.

i.e., if $|x(n)| < \infty$, then $|y(n)| < \infty$, for all values of n.

Examples:

- $y(n) = T[x(n)] = 2x(n)+3$
- $y(n) = T[x(n)] = 2x(-n-3)+3x^2(n)$
- $y(n) = T[x(n)] = 3Cos[x(n)]$
- $y(n) = T[x(n)] = -\sum_{k=1}^{N} a_K\, y(n+k) + \sum_{k=0}^{M} b_K\, x(n-k)$
- $y(n) = T[x(n)] = \sum_{k=-4}^{\infty} b_K\, x(n-k)$

} Stable Systems

- $y(n) = T[x(n)] = nx(n)$
- $y(n) = T[x(n)] = \dfrac{2}{x(n)}$

} Untable Systems

- If n=∞, then y(n) = T[x(n)] = nx(n) = ∞ = unbounded, hence the system is unstable.
- If x(n)=0, then $y(n) = T[x(n)] = \dfrac{2}{x(n)}$ = ∞ = unbounded, hence the system is unstable.

Note: Impulse response h(n) of stable system is absolutely summable, $\sum_{k=-4}^{\infty} |h(n)| < \infty$

Example-1: Test the discrete time system $h(n)=(1/2)^n\, u(n)$ for stability.

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} \left|\left(\frac{1}{2}\right)^n u(n)\right| = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\ldots\ldots = \frac{1}{1-\frac{1}{2}} = 2 = Finite$$

Given causal system is stable, because the impulse response is absolutely summable.

Example-2: Test the discrete time system $h(n)=2^n\, u(n)$ for stability.

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} |2^n u(n)| = \sum_{n=0}^{\infty} 2^n = 1+2+4+8+\ldots.. = Infinity$$

Given causal system is unstable, because the impulse response is not absolutely summable.

Example-3: Test the discrete time system h(n)=2^n u(-n) for stability.

$$\sum_{n=-\infty}^{\infty} | h(n) | = \sum_{n=-\infty}^{\infty} \left| 2^n u(-n) \right| = \sum_{n=-\infty}^{0} 2^n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + ... = \frac{1}{1 - 1/2} = 2 = Finite$$

Given noncausal system is stable, because the impulse response is absolutely summable.

Example-4: Test the discrete time system h(n)=$(1/2)^n$ u(-n) for stability.

$$\sum_{n=-\infty}^{\infty} | h(n) | = \sum_{n=-\infty}^{\infty} \left| (1/2)^n u(-n) \right| = \sum_{n=-\infty}^{0} (1/2)^n = 1 + 2 + 4 + 8 + ... = Infinity$$

Given noncausal system is unstable, because the impulse response is not absolutely summable.

Discrete Time Fourier Transform (DTFT):

Fourier Transform (FT) is a mathematical tool, which is used to compute the frequency domain signal form the time domain signal and vice versa. Fourier Transform is used for both the continuous time and discrete time signals. Fourier Transform of continuous time signal is called Continuous Time Fourier Transform (CTFT) or simply Fourier Transform

Fourier Transform of a discrete time signal is called Discrete Time Fourier Transform. The DTFT of a signal x(n) is represented with $X(e^{j\omega})$ and it can be computed from the formula.

$$DTFT[\ x(n)\ ] = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

In general, the frequency domain $X(e^{j\omega})$ is in complex form and it can be expressed as

$$X(e^{j\omega}) = X_R\,(e^{j\omega}) + j\, X_I\,(e^{j\omega})$$

Where,

$X_R\,(e^{j\omega})$: Real part of $X\,(e^{j\omega})$

$X_I\,(e^{j\omega})$: Imaginary part of $X\,(e^{j\omega})$

Magnitude of $X(e^{j\omega})$ is called the magnitude spectrum and it can be computed from the formula

$$\left|X(e^{j\omega})\right| = \sqrt{[\,X_R\,(e^{j\omega})\,]^2 + [\,X_I\,(e^{j\omega})\,]^2}$$

Phase of $X(e^{j\omega})$ is called the phase spectrum and it can be computed from the formula

$$\angle X\,(e^{j\omega}) = Tan^{-1}\left(\frac{X_I\,(e^{j\omega})}{X_R\,(e^{j\omega})}\right)$$

Discrete time signal x(n) can be computed from the frequency domain $X\,(e^{j\omega})$ is called Inverse Discrete Time Fourier Transform (IDTFT).

$$x(n) = IDTFT[X(e^{j\omega})] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

Discrete time signal x(n) and the frequency domain $X\,(e^{j\omega})$ are Discrete Time Fourier Transformable pairs.

$$x(n) \underset{IDTFT}{\overset{DTFT}{\rightleftarrows}} X(e^{j\omega})$$

Example-1: Determine the DTFT of a sequence $x(n)=a^n u(n)$, hence obtain the magnitude and phase spectrum.

$$\text{DTFT}[\,x(n)\,] = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^n = 1 + ae^{-j\omega} + \left(ae^{-j\omega}\right)^2 + \ldots\ldots\ldots. = \frac{1}{1-ae^{-j\omega}}$$

$$X(e^{j\omega}) = \frac{1}{1-a(Cos\omega - jSin\omega)} = \frac{1}{1-aCos\omega + jaSin\omega}$$

Magnitude spectrum $\left|X(e^{j\omega})\right| = \sqrt{[\,X_R\,(e^{j\omega})\,]^2 + [\,X_I\,(e^{j\omega})\,]^2}$

$$\left|X(e^{j\omega})\right| = \left|\frac{1}{1-aCos\omega + jaSin\omega}\right| = \frac{1}{\sqrt{(1-aCos\omega)^2 + (aSin\omega)^2}} = \frac{1}{\sqrt{1+a^2Cos^2\omega - 2aCos\omega + a^2Sin^2\omega}}$$

$$\left|X(e^{j\omega})\right| = \frac{1}{\sqrt{1+a^2-2aCos\omega}}$$

Phase spectrum $\angle X\,(e^{j\omega}) = Tan^{-1}\left(\frac{X_I\,(e^{j\omega})}{X_R\,(e^{j\omega})}\right)$

$$\angle X(e^{j\omega}) = \angle(1+j0) - \angle(1-aCos\omega + jaSin\omega) = Tan^{-1}\left(\frac{0}{1}\right) - Tan^{-1}\left(\frac{aSin\omega}{1-aCos\omega}\right) = -Tan^{-1}\left(\frac{aSin\omega}{1-aCos\omega}\right)$$

Example-2: Determine the DTFT of a sequence $x(n)=(1/2)^n u(n)$, hence obtain the magnitude and phase spectrum.

$$\text{DTFT}[\,x(n)\,] = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n) e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}e^{-j\omega}\right)^n = 1 + \frac{1}{2}e^{-j\omega} + \left(\frac{1}{2}e^{-j\omega}\right)^2 + \ldots\ldots\ldots.$$

$$X(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}} = \frac{2}{2-e^{-j\omega}} = \frac{2}{2-(Cos\omega - jSin\omega)} = \frac{2}{2-Cos\omega + jSin\omega}$$

Magnitude spectrum $\left|X(e^{j\omega})\right| = \sqrt{[\,X_R\,(e^{j\omega})\,]^2 + [\,X_I\,(e^{j\omega})\,]^2}$

$$\left|X(e^{j\omega})\right| = \left|\frac{2}{2-Cos\omega + jSin\omega}\right| = \frac{2}{\sqrt{(2-Cos\omega)^2 + (Sin\omega)^2}} = \frac{2}{\sqrt{4+Cos^2\omega - 4Cos\omega + Sin^2\omega}} = \frac{2}{\sqrt{5-4Cos\omega}}$$

Phase spectrum $\angle X\,(e^{j\omega}) = Tan^{-1}\left(\frac{X_I\,(e^{j\omega})}{X_R\,(e^{j\omega})}\right)$

$$\angle X(e^{j\omega}) = \angle(2+j0) - \angle(2-Cos\omega + jSin\omega) = Tan^{-1}\left(\frac{0}{2}\right) - Tan^{-1}\left(\frac{Sin\omega}{2-Cos\omega}\right) = -Tan^{-1}\left(\frac{Sin\omega}{2-Cos\omega}\right)$$

Example-3: Determine the DTFT of a sequence x(n)=a^n u(-n), hence obtain the magnitude and phase spectrum.

$$DTFT[\, x(n)\,] = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n\, u(-n) e^{-j\omega n} = \sum_{n=-\infty}^{0} a^n e^{-j\omega n} = \sum_{n=-\infty}^{0} \left(\frac{1}{a} e^{j\omega}\right)^{-n} = 1 + \frac{1}{a} e^{j\omega} + \left(\frac{1}{a} e^{j\omega}\right)^2 + ...$$

$$X(e^{j\omega}) = \frac{1}{1-\frac{1}{a}e^{j\omega}} = \frac{a}{a-e^{j\omega}} = \frac{a}{a-(Cos\omega + jSin\omega)} = \frac{a}{a - Cos\omega - jSin\omega}$$

Magnitude spectrum $\left|X(e^{j\omega})\right| = \sqrt{[\, X_R\,(e^{j\omega})\,]^2 + [\, X_I\,(e^{j\omega})\,]^2}$

$$\left|X(e^{j\omega})\right| = \left|\frac{a}{a - Cos\omega - jSin\omega}\right| = \frac{a}{\sqrt{(a-Cos\omega)^2 + (-Sin\omega)^2}} = \frac{a}{\sqrt{a^2 + Cos^2\omega - 2aCos\omega + Sin^2\omega}}$$

$$\left|X(e^{j\omega})\right| = \frac{a}{\sqrt{1 + a^2 - 2aCos\omega}}$$

Phase spectrum $\angle X\,(e^{j\omega}) = Tan^{-1}\left(\frac{X_I\,(e^{j\omega})}{X_R\,(e^{j\omega})}\right)$

$$\angle X(e^{j\omega}) = \angle(a + j0) - \angle(a - Cos\omega - jSin\omega) = Tan^{-1}\left(\frac{0}{a}\right) - Tan^{-1}\left(\frac{-Sin\omega}{a - Cos\omega}\right) = Tan^{-1}\left(\frac{Sin\omega}{a - Cos\omega}\right)$$

Example-4: Determine the DTFT of a sequence x(n)=2^n u(-n), hence obtain the magnitude and phase spectrum.

$$DTFT[\, x(n)\,] = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} 2^n\, u(-n) e^{-j\omega n} = \sum_{n=-\infty}^{0} 2^n e^{-j\omega n} = \sum_{n=-\infty}^{0} \left(\frac{1}{2} e^{j\omega}\right)^{-n} = 1 + \frac{1}{2} e^{j\omega} + \left(\frac{1}{2} e^{j\omega}\right)^2 + ...$$

$$X(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{j\omega}} = \frac{2}{2-e^{j\omega}} = \frac{2}{2-(Cos\omega + jSin\omega)} = \frac{2}{2 - Cos\omega - jSin\omega}$$

Magnitude spectrum $\left|X(e^{j\omega})\right| = \sqrt{[\, X_R\,(e^{j\omega})\,]^2 + [\, X_I\,(e^{j\omega})\,]^2}$

$$\left|X(e^{j\omega})\right| = \left|\frac{2}{2 - Cos\omega - jSin\omega}\right| = \frac{2}{\sqrt{(2-Cos\omega)^2 + (-Sin\omega)^2}} = \frac{2}{\sqrt{4 + Cos^2\omega - 4Cos\omega + Sin^2\omega}} = \frac{2}{\sqrt{5 - 4Cos\omega}}$$

Phase spectrum $\angle X\,(e^{j\omega}) = Tan^{-1}\left(\frac{X_I\,(e^{j\omega})}{X_R\,(e^{j\omega})}\right)$

$$\angle X(e^{j\omega}) = \angle(2 + j0) - \angle(2 - Cos\omega - jSin\omega) = Tan^{-1}\left(\frac{0}{2}\right) - Tan^{-1}\left(\frac{-Sin\omega}{2 - Cos\omega}\right) = Tan^{-1}\left(\frac{Sin\omega}{2 - Cos\omega}\right)$$

Example-5: Determine the DTFT of a sequence $x(n) = a^{|n|}$, hence obtain the magnitude and phase spectrum.

$$DTFT[\ x(n)\ ] = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\ e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\omega n} = \sum_{n=-1}^{-\infty} \left(ae^{j\omega}\right)^{-n} + \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^{n}$$

$$X(e^{j\omega}) = ae^{j\omega} + \left(ae^{j\omega}\right)^2 + ...+1 + ae^{-j\omega} + \left(ae^{-j\omega}\right)^2 + ...$$

$$X(e^{j\omega}) = -1 + 1 + ae^{j\omega} + \left(ae^{j\omega}\right)^2 + ...+1 + ae^{-j\omega} + \left(\frac{1}{2}e^{-j\omega}\right)^2 + ...$$

$$X(e^{j\omega}) = -1 + \frac{1}{1-ae^{j\omega}} + \frac{1}{1-ae^{-j\omega}} = -1 + \frac{(1-ae^{-j\omega}+1-ae^{j\omega})}{(1-ae^{j\omega})(1-ae^{-j\omega})} = -1 + \frac{2-a(e^{j\omega}+e^{-j\omega})}{1+a^2-a(e^{j\omega}+e^{-j\omega})}$$

$$X(e^{j\omega}) = -1 + \frac{2-2aCos\omega}{1+a^2-2aCos\omega} = \frac{-(1+a^2-2aCos\omega)+2-2aCos\omega}{1+a^2-2aCos\omega} = \frac{1-a^2}{1+a^2-2aCos\omega}$$

Example-6: Determine the DTFT of a sequence $x(n) = \left(\frac{1}{2}\right)^{|n|}$, hence obtain the magnitude and phase spectrum.

$$DTFT[\ x(n)\ ] = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\ e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n|} e^{-j\omega j} = \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} e^{-j\omega j} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} e^{-j\omega j} = \sum_{n=-1}^{-\infty} \left(\frac{1}{2}e^{j\omega}\right)^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}e^{-j\omega}\right)^{n}$$

$$X(e^{j\omega}) = \frac{1}{2}e^{j\omega} + \left(\frac{1}{2}e^{j\omega}\right)^2 + ...+1 + \frac{1}{2}e^{-j\omega} + \left(\frac{1}{2}e^{-j\omega}\right)^2 + ...$$

$$X(e^{j\omega}) = -1 + 1 + \frac{1}{2}e^{j\omega} + \left(\frac{1}{2}e^{j\omega}\right)^2 + ...+1 + \frac{1}{2}e^{-j\omega} + \left(\frac{1}{2}e^{-j\omega}\right)^2 + ...$$

$$X(e^{j\omega}) = -1 + \frac{1}{1-\frac{1}{2}e^{j\omega}} + \frac{1}{1-\frac{1}{2}e^{-j\omega}} = -1 + \frac{2}{2-e^{j\omega}} + \frac{2}{2-e^{-j\omega}} = -1 + \frac{2(2-e^{-j\omega}+2-e^{j\omega})}{(2-e^{j\omega})(2-e^{-j\omega})}$$

$$X(e^{j\omega}) = -1 + \frac{2(4-2Cos\omega)}{(4-4Cos\omega+1)} = -1 + \frac{8-4Cos\omega}{5-4Cos\omega} = \frac{-5+4Cos\omega+8-4Cos\omega}{5-4Cos\omega} = \frac{3}{5-4Cos\omega}$$

Example-7: Determine DTFT of a sequence x(n)=δ(n).

$$DTFT[\ x(n)\ ] = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\ e^{-j\omega n}$$

$$DTFT[\ \delta(n)\ ] = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta(n)\ e^{-j\omega n} = ...+\delta(0)e^{-j0} + ... = 1\times 1 = 1$$

Example-8: Determine the sequence x(n), given $X(e^{j\omega}) = 1$, $-\pi/N \le \omega \le \pi/N$.

$$x(n) = IDTFT[X(e^{j\omega})] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi/N}^{\pi/N} e^{j\omega n}\, d\omega = \frac{1}{2\pi}\frac{e^{j\omega n}}{jn}\Bigg|_{-\pi/N}^{\pi/N} = \frac{1}{2\pi}\frac{e^{jn\pi/N} - e^{-jn\pi/N}}{jn}$$

$$x(n) = \frac{1}{2\pi}\frac{2jSin(n\pi/N)}{jn} = \frac{Sin(n\pi/N)}{n\pi}$$

Example-9: Determine the sequence x(n), given $X(e^{j\omega}) = 1+\cos\omega+2\cos2\omega+3\cos3\omega$.

$$X(e^{j\omega}) = 1 + \frac{e^{jw}+e^{-jw}}{2} + 2\frac{e^{j2w}+e^{-j2w}}{2} + 3\frac{e^{j3w}+e^{-j3w}}{2}$$

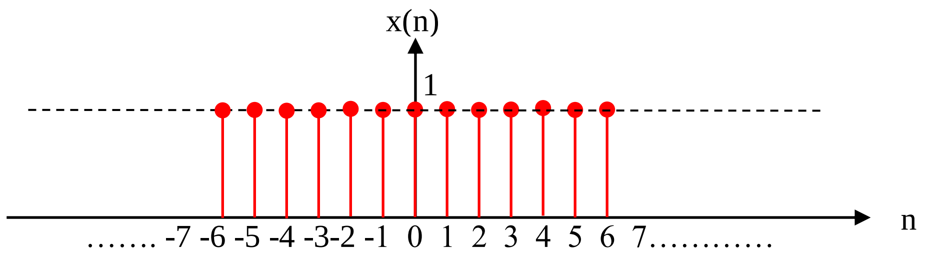
$$X(e^{j\omega}) = 1 + \frac{1}{2}e^{jw} + \frac{1}{2}e^{-jw} + e^{j2w} + e^{-j2w} + \frac{3}{2}e^{j3w} + \frac{3}{2}e^{-j3w}$$

$$X(e^{j\omega}) = \frac{3}{2}e^{j3w} + e^{j2w} + \frac{1}{2}e^{jw} + 1 + \frac{1}{2}e^{-jw} + e^{-j2w} + \frac{3}{2}e^{-j3w}$$

$$\sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn} = x(-3)e^{j3w} + x(-2)e^{j2w} + x(-1)e^{jw} + x(0) + x(1)e^{-jw} + x(2)e^{-j2w} + x(3)e^{-j3w}$$

$$\Rightarrow x(n) = \left\{\frac{3}{2}, 1, \frac{1}{2}, \underset{\uparrow}{1}, \frac{1}{2}, 1, \frac{3}{2}\right\}$$

Example-10: Determine the DTFT of a periodic sequence $x(n) = 1$.



$$DTFT[\ x(n)\ ] = X(e^{j\omega}) = 2\pi\sum_{m=-\infty}^{\infty}\delta(\omega - 2\pi m)$$

Properties of DTFT:

Various properties used in the Discrete Time Fourier Transform (DTFT) are given below

- Linear Property
- Periodicity or Periodic Property
- Time Shifting Property
- Frequency Shifting Property
- Time Reversal Property
- Conjugation or Conjugate Property
- Frequency Differentiation Property
- Time Convolution Theorem
- Frequency Convolution Theorem
- Parsevalls Theorem

(A)Linear Property:

If $x_1(n)$, $x_2(n)$ are two discrete time signals and DTFT[$x_1(n)$] = $X_1(e^{j\omega})$, DTFT[$x_2(n)$] = $X_2(e^{j\omega})$, then DTFT[a $x_1(n)$ + b $x_2(n)$] = a $X_1(e^{j\omega})$ + b $X_2(e^{j\omega})$ is called linear property of DTFT.

Proof:

From the basic definition of DTFT

$$\text{DTFT}[\, x(n)\,] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$\text{Replace } \; x(n) \; \text{by} \; a\, x_1(n) + b\, x_2(n)$$

$$\begin{aligned}
\text{DTFT}[a\; x_1(n) + b\, x_2(n)] &= \sum_{n=-\infty}^{\infty} [a\, x_1(n) + b\, x_2(n)]\, e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} [a\, x_1(n) e^{-j\omega n} + b\, x_2(n)\, e^{-j\omega n}] \\
&= \sum_{n=-\infty}^{\infty} [a\, x_1(n) e^{-j\omega n}] + \sum_{n=-\infty}^{\infty} [b\, x_2(n)\, e^{-j\omega n}] \\
&= a \sum_{n=-\infty}^{\infty} [\, x_1(n) e^{-j\omega n}] + b \sum_{n=-\infty}^{\infty} [\, x_2(n)\, e^{-j\omega n}] \\
&= a\, X_1(e^{j\omega}) + b\, X_2(e^{j\omega})
\end{aligned}$$

Example : Determine the DTFT of a sequence $x(n)= a.b^n u(n)+b.a^n u(n)$.

$$\text{DTFT}[x(n)] = DTFT[a.b^n u(n) + b.a^n u(n)] = aDTFT[b^n u(n)] + bDTFT[a^n u(n)]$$

$$X(e^{j\omega}) = \frac{a}{1-be^{-j\omega}} + \frac{b}{1-ae^{-j\omega}} = \frac{a(1-ae^{-j\omega}) + b(1-be^{-j\omega})}{(1-be^{-j\omega})(1-ae^{-j\omega})} = \frac{a+b-(a^2+b^2)e^{-j\omega}}{1-(a+b)e^{-j\omega}+abe^{-j2\omega}}$$

(B)Periodicity or Periodic Property:

If x(n) is a discrete time signal and DTFT[x(n)] = $X(e^{j\omega})$, then $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$, i.e the frequency domain $X(e^{j\omega})$ is periodic with a period of '2π' irrespective of x(n).

Proof:

From the basic definition of DTFT

$$DTFT[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Replace ω by ω + 2π

$$X(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega+2\pi)n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\, e^{-j2\pi n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\,(1)$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= X(e^{j\omega})$$

Example: Determine the DTFT of a real and even sequence $x(n) = \left\{3,2,\underset{\uparrow}{1},2,3\right\}$, hence obtain the period of $X(e^{j\omega})$.

$$DTFT[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \sum_{n=-2}^{2} x(n)\, e^{-j\omega n} = x(-2)\, e^{j2\omega} x(-1)\, e^{j\omega} + x(0)\, e^{-j0} + x(1)\, e^{-j\omega} + x(2)\, e^{-j2\omega}$$

$$X(e^{j\omega}) = 3e^{j2\omega} + 2e^{j\omega} + 1 + 2e^{-j\omega} + 3e^{-j2\omega}$$

$$X(e^{j\omega}) = 1 + 2(e^{j\omega} + e^{-j\omega}) + 3(e^{j2\omega} + e^{-j2\omega})$$

$$X(e^{j\omega}) = 1 + 2(2Cos\omega) + 3(2Cos2\omega)$$

$$X(e^{j\omega}) = 1 + 4Cos\omega + 6Cos2\omega$$

$$\Rightarrow X(e^{j(\omega+2\pi)}) = 1 + 4Cos(\omega+2\pi) + 6Cos2(\omega+2\pi)$$

$$\Rightarrow X(e^{j(\omega+2\pi)}) = 1 + 4Cos(2\pi+\omega) + 6Cos(4\pi+2\omega)$$

$$\Rightarrow X(e^{j(\omega+2\pi)}) = 1 + 4Cos(\omega) + 6Cos(2\omega)$$

$$\Rightarrow X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

Period of $X(e^{j\omega})$ is 2π.

(C)Time Shifting Property:

If x(n) is a discrete time signal and DTFT[x(n)] = $X(e^{j\omega})$, then DTFT[$x(n - n_0)$] = $e^{-j\omega n_0} X(e^{j\omega})$ is called time shifting property of DTFT.

Proof:

From basic definition of DTFT

$$\text{DTFT}[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Replace x(n) by $x(n - n_0)$

$$\text{DTFT}[\,x(n-n_0)\,] = \sum_{n=-\infty}^{\infty} x(n-n_0)\, e^{-j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega (no+m)}$$

$$= \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega no} e^{-j\omega m}$$

$$= e^{-j\omega no} \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega m}$$

$$= e^{-j\omega no}\, X(e^{j\omega})$$

Let $n - n_0 = m$
Then $n = n_0 + m$

Range of summation
LL (m) = $-\infty - n_0 = -\infty$
UL (m) = $\infty - n_0 = \infty$

Example : Determine the DTFT of a sequence y(n)=x(n-1), given $x(n)= a.b^n u(n)+b.a^n u(n)$.

$$\text{DTFT[x(n)]} = DTFT[a.b^n u(n) + b.a^n u(n)] = aDTFT[b^n u(n)] + bDTFT[a^n u(n)]$$

$$X(e^{j\omega}) = \frac{a}{1-be^{-j\omega}} + \frac{b}{1-ae^{-j\omega}} = \frac{a(1-ae^{-j\omega}) + b(1-be^{-j\omega})}{(1-be^{-j\omega})(1-ae^{-j\omega})} = \frac{a+b-(a^2+b^2)e^{-j\omega}}{1-(a+b)e^{-j\omega}+abe^{-j2\omega}}$$

Apply time shifting property DTFT[$x(n - n_0)$] = $e^{-j\omega n_0} X(e^{j\omega})$

$\Rightarrow$DTFT[x(n – 1)] = DTFT[y(n)]=$e^{-j\omega} X(e^{j\omega})$

$$\Rightarrow Y(e^{j\omega}) = e^{-j\omega}\, \frac{a+b-(a^2+b^2)e^{-j\omega}}{1-(a+b)e^{-j\omega}+abe^{-j2\omega}}$$

$$\Rightarrow Y(e^{j\omega}) = \frac{(a+b)e^{-j\omega} - (a^2+b^2)e^{-j2\omega}}{1-(a+b)e^{-j\omega}+abe^{-j2\omega}}$$

(D)Frequency Shifting Property:

If x(n) is a discrete time signal and DTFT[x(n)] = $X(e^{j\omega})$, then DTFT[$e^{j\omega_0 n}$ x(n)] = $X(e^{j(\omega - \omega_0)})$ is called frequency shifting property of DTFT.

Proof:

From the basic definition of DTFT

$$DTFT[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Replace x(n) by $e^{j\omega o n}$ x(n)

$$DTFT[\,e^{j\omega o n} x(n)\,] = \sum_{n=-\infty}^{\infty} e^{j\omega o n} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{j\omega o n} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega - \omega o) n}$$

$$= X(e^{j(\omega - \omega o)})$$

(E)Time Reversal Property:

If x(n) is a discrete time signal and DTFT[x(n)] = $X(e^{j\omega})$, then DTFT[x(–n)] = $X(e^{-j\omega})$ is called time reversal property of DTFT.

Proof:

From the basic definition of DTFT

$$DTFT[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Replace x(n) by x(–n)

$$DTFT[\,x(-n)\,] = \sum_{n=-\infty}^{\infty} x(-n)\, e^{-j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega(-n)}$$

$$= \sum_{m=\infty}^{-\infty} x(m)\, e^{j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} x(m)\, e^{-j(-\omega) n}$$

$$= X(e^{-j\omega})$$

Let – n = m
Then n = - m

Range of summation
LL (m) = –(– ∞) = ∞
UL (m) = –(∞) = – ∞

(F)Conjugation or Conjugate Property:

If x(n) is a discrete time signal and DTFT[x(n)] = $X(e^{j\omega})$, then DTFT[x*(n)] = $X^*(e^{-j\omega})$ or DTFT[x*(–n)] = $X^*(e^{j\omega})$ is called conjugate property of DTFT.

Proof:

From the basic definition of DTFT

$$\text{DTFT}[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Replace x(n) by x*(n)

$$\text{DTFT}[\,x^*(n)\,] = \sum_{n=-\infty}^{\infty} x^*(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x^*(n)\,[e^{j\omega n}]^*$$

$$= \sum_{n=-\infty}^{\infty} [\,x(n)\; e^{j\omega n}]^*$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n)\; e^{j\omega n}\right]^*$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n)\; e^{-j(-\omega)n}\right]^*$$

$$= \left[\sum_{n=-\infty}^{\infty} X(\,e^{-j\omega}\,)\right]^*$$

$$= [X(e^{-j\omega}\,)]^*$$

$$= X^*(e^{-j\omega}\,)$$

Note : Apply time reversal property on DTFT[x*(n)] = $X^*(e^{-j\omega})$ ⇒ DTFT[x*(–n)] = $X^*(e^{j\omega})$

Example-1: Determine the DTFT of a real sequence $x(n) = \left\{1, \underset{\uparrow}{2}, 3, 4\right\}$

$$\text{DTFT}[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \sum_{n=-1}^{2} x(n)\, e^{-j\omega n} = x(-1)\, e^{j\omega} + x(0)\, e^{-j0} + x(1)\, e^{-j\omega} + x(2)\, e^{-j2\omega}$$

$$X(e^{j\omega}) = e^{j\omega} + 2 + 3\, e^{-j\omega} + 4e^{-j2\omega}$$

$$X(e^{j\omega}) = Cos\omega + jSin\omega + 2 + 3(Cos\omega - jSin\omega) + 4(Cos2\omega - jSin2\omega)$$

$$X(e^{j\omega}) = 2 + Cos\omega + 3Cos\omega + 4Cos2\omega + jSin\omega - j3Sin\omega - j4Sin2\omega$$

$$X(e^{j\omega}) = 2 + 4(Cos\omega + Cos2\omega) - j2(Sin\omega + 2Sin2\omega)$$

Note:

- If x(n) is real, then $X^*(e^{-j\omega}) = X(e^{j\omega})$, i.e. $X(e^{j\omega})$ and $X(e^{-j\omega})$ are complex conjugate pairs.

  $$X(e^{-j\omega}) = 2 + 4Cos\omega + 4Cos2\omega + j2(Sin\omega + 2Sin2\omega)$$

  $$X^*(e^{-j\omega}) = 2 + 4Cos\omega + 4Cos2\omega - j2(Sin\omega + 2Sin2\omega) = X(e^{j\omega})$$

- If x(n) is real and $X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$, then $X_R(e^{j\omega})$ is even and $X_I(e^{j\omega})$ is odd.

$$X(e^{j\omega}) = 2 + 4Cos\omega + 4Cos2\omega - j2(Sin\omega + 2Sin2\omega) = X_R(e^{j\omega}) + X_I(e^{j\omega})$$

$$X_R(e^{j\omega}) = 2 + 4(Cos\omega + Cos2\omega) = EvenFunction$$

$$X_I(e^{j\omega}) = -2(Sin\omega + 2Sin2\omega) = OddFunction$$

- If x(n) is real, then $|X(e^{j\omega})| = |X(e^{-j\omega})|$, i.e Magnitude spectrum is even and $\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$, i.e Phase spectrum is odd

$$X(e^{j\omega}) = 2 + 4Cos\omega + 4Cos2\omega - j2(Sin\omega + 2Sin2\omega) = X_R(e^{j\omega}) + X_I(e^{j\omega})$$

$$X(e^{-j\omega}) = 2 + 4Cos\omega + 4Cos2\omega + j2(Sin\omega + 2Sin2\omega) = X_R(e^{j\omega}) - X_I(e^{j\omega})$$

Example-2: Determine the DTFT of a real and even sequence $x(n) = \left\{3,2,\underset{\uparrow}{1},2,3\right\}$

$$DTFT[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n)\,e^{-j\omega n} = \sum_{n=-2}^{2} x(n)\,e^{-j\omega n} = x(-2)\,e^{j2\omega} x(-1)\,e^{j\omega} + x(0)\,e^{-j0} + x(1)\,e^{-j\omega} + x(2)\,e^{-j2\omega}$$

$$X(e^{j\omega}) = 3e^{j2\omega} + 2e^{j\omega} + 1 + 2e^{-j\omega} + 3e^{-j2\omega}$$

$$X(e^{j\omega}) = 1 + 2(e^{j\omega} + e^{-j\omega}) + 3(e^{j2\omega} + e^{-j2\omega})$$

$$X(e^{j\omega}) = 1 + 2(2Cos\omega) + 3(2Cos2\omega)$$

$$X(e^{j\omega}) = 1 + 4Cos\omega + 6Cos2\omega$$

Note: If x(n) is real and even, then $X(e^{j\omega})$ is real and even.

Example-3: Determine the DTFT of a real and odd sequence $x(n) = \left\{3,2,\underset{\uparrow}{0},-2,-3\right\}$

$$DTFT[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n)\,e^{-j\omega n} = \sum_{n=-2}^{2} x(n)\,e^{-j\omega n} = x(-2)\,e^{j2\omega} x(-1)\,e^{j\omega} + x(0)\,e^{-j0} + x(1)\,e^{-j\omega} + x(2)\,e^{-j2\omega}$$

$$X(e^{j\omega}) = 3e^{j2\omega} + 2e^{j\omega} + 0 - 2e^{-j\omega} - 3e^{-j2\omega}$$

$$X(e^{j\omega}) = 2(e^{j\omega} - e^{-j\omega}) + 3(e^{j2\omega} - e^{-j2\omega})$$

$$X(e^{j\omega}) = 2(2jSin\omega) + 3(2jSin2\omega)$$

$$X(e^{j\omega}) = j2(2Sin\omega + 3Sin2\omega)$$

Note: If x(n) is real and odd, then $X(e^{j\omega})$ is imaginary and odd.

(G)Frequency Differentiation Property:

If x(n) is a discrete time signal and DTFT[x(n)] = $X(e^{j\omega})$, then $DTFT[nx(n)] = j\frac{d}{d\omega}\left[X(e^{j\omega})\right]$ is called frequency differentiation property of DTFT.

Proof:

From the basic definition of DTFT

$$DTFT[\,x(n)\,] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Differentiate w.r.t. ω

$$\frac{d}{d\omega}\left[X(e^{j\omega})\right] = \sum_{n=-\infty}^{\infty} x(n)\, \frac{d}{d\omega} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\,(-jn) e^{-j\omega n}$$

$$= -j \sum_{n=-\infty}^{\infty} nx(n)\, e^{-j\omega n}$$

$$= -jDTFT[nx(n)]$$

$$DTFT[nx(n)] = j\frac{d}{d\omega}\left[X(e^{j\omega})\right]$$

Example: Determine the DTFT of a sequence $x(n) = na^n u(n)$

We know that $DTFT[\,a^n u(n)] = \frac{1}{1-ae^{-j\omega}}$

Apply the frequency differentiation property $DTFT[nx(n)] = j\frac{d}{d\omega}\left[X(e^{j\omega})\right]$

$$DTFT[na^n u(n)] = j\frac{d}{d\omega}\frac{1}{1-ae^{-j\omega}} = j\frac{-1}{\left(1-ae^{-j\omega}\right)^2}\left(0-a(-j)e^{-j\omega}\right)$$

$$DTFT[na^n u(n)] = \frac{-j}{\left(1-ae^{-j\omega}\right)^2}\left(jae^{-j\omega}\right)$$

$$= \frac{ae^{-j\omega}}{\left(1-ae^{-j\omega}\right)^2}$$

(H)Time Convolution Theorem:

If $x_1(n)$, $x_2(n)$ are two discrete time signals and DTFT[$x_1(n)$] = $X_1(e^{j\omega})$, DTFT[$x_2(n)$] = $X_2(e^{j\omega})$, then DTFT[$x_1(n) * x_2(n)$] = $X_1(e^{j\omega})\, X_2(e^{j\omega})$ is called time convolution theorem of DTFT.

i.e, convolution in time domain leads to multiplication in frequency domain.

Proof:

From the basic definition of DTFT

$$DTFT[\, x(n)\,] = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Replace x(n) with $x_1(n) * x_2(n)$

$$DTFT[\, x_1(n)*x_2(n)\,] = \sum_{n=-\infty}^{\infty} x_1(n)*x_2(n)\, e^{-j\omega n}$$

Apply convolution formula $x_1(n)*x_2(n) = \sum_{m=-\infty}^{\infty} x_1(m)x_2(n-m)$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x_1(m)x_2(n-m)\, e^{-j\omega n}$$

Change the order of summation

$$= \sum_{m=-\infty}^{\infty} x_1(m) \sum_{n=-\infty}^{\infty} x_2(n-m)\, e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x_1(m)\, DTFT[x_2(n-m)]$$

$$= \sum_{m=-\infty}^{\infty} x_1(m)\, e^{-j\omega m} X_2(e^{j\omega}) = X_2(e^{j\omega}) \sum_{m=-\infty}^{\infty} x_1(m)\, e^{-j\omega m}$$

$$= X_2(e^{j\omega})X_1(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega})$$

Example: Determine the DTFT of a sequence $x(n) = a^n u(n) * n a^n u(n)$

We know that $DTFT[a^n u(n)] = \frac{1}{1-ae^{-j\omega}}$ and $DTFT[na^n u(n)] = \frac{ae^{-j\omega}}{\left(1-ae^{-j\omega}\right)^2}$

$$DTFT[a^n u(n) * na^n u(n)] = DTFT[a^n u(n)]DTFT[na^n u(n)]$$

$$= \frac{1}{\left(1-ae^{-j\omega}\right)} \frac{ae^{-j\omega}}{\left(1-ae^{-j\omega}\right)^2} = \frac{ae^{-j\omega}}{\left(1-ae^{-j\omega}\right)^3}$$

(I)Frequency Convolution Theorem:

If $x_1(n)$, $x_2(n)$ are two discrete time signals and DTFT[$x_1(n)$] = $X_1(e^{j\omega})$, DTFT[$x_2(n)$] = $X_2(e^{j\omega})$, then 2π DTFT[$x_1(n)\, x_2(n)$] = $X_1(e^{j\omega}) * X_2(e^{j\omega})$ is called frequency convolution theorem of DTFT.

i.e, convolution in frequency domain leads to multiplication in time domain.

(J)Parsevalls Theorem:

If x(n) is a discrete time signal and DTFT[x(n)] = $X(e^{j\omega})$, then the total energy under the signal x(n) can be computed from x(n) as well as $X(e^{j\omega})$ through the Parsevalls theorem

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega$$

Proof:

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x(n)\; x^*(n) = \sum_{n=-\infty}^{\infty} x(n)\; [x(n)]^*$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega\right]^* = \sum_{n=-\infty}^{\infty} x(n) \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega}) e^{-j\omega n} d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega}) \left(\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\right) d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega}) \left(X(e^{j\omega})\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega$$

Example:

Evaluate (i) $X(e^{j\omega})$ at ω = 0 (ii) $\int_{-\pi}^{\pi} X(e^{j\omega})\, d\omega$ (iii) $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega$

Given that DTFT[x(n)] = $X(e^{j\omega})$ and $x(n) = \{1,4,\underset{\uparrow}{3},2,5,6,7,0,9\}$.

(i) $\text{DTFT}[\,x(n)\,] = X(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$

$$X(e^{j\omega})\big|_{w-0} = \sum_{n=-\infty}^{\infty} x(n) = 1+4+3+2+5+6+7+0+9 = 37$$

(ii) $x(n) = IDTFT[X(e^{j\omega})] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n} d\omega$

Put n = 0 $\Rightarrow x(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) d\omega \Rightarrow \int_{-\pi}^{\pi} X(e^{j\omega}) d\omega = 2\pi x(0) = 2\pi(3) = 6\pi$

(ii)From Parsevalls theorem

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega$$

$$\Rightarrow \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega = 2\pi \sum_{n=-\infty}^{\infty} |x(n)|^2 = 2\pi\left(1^2+4^2+3^2+2^2+5^2+6^2+7^2+0^2+9^2\right) = 442\pi$$

Analysis of Discrete LSI Systems Using DTFT:

Frequency Response of Discrete LSI System:

The ratio between output y(n) to input x(n) in frequency domain representation is called Transfer function or System function or Frequency response of discrete LSI System and it is represented with $H(e^{j\omega})$.

$$H(e^{j\omega}) = \frac{DTFT[y(n)]}{DTFT[x(n)]} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

In general, the frequency response $H(e^{j\omega})$ is in complex form and it can be expressed as

$$H(e^{j\omega}) = H_R(e^{j\omega}) + j\,H_I(e^{j\omega})$$

Where,

$H_R(e^{j\omega})$: Real part of $H(e^{j\omega})$

$H_I(e^{j\omega})$: Imaginary part of $H(e^{j\omega})$

Magnitude of $H(e^{j\omega})$ is called the magnitude response and it can be computed from the formula

$$|H(e^{j\omega})| = \sqrt{[H_R(e^{j\omega})]^2 + [H_I(e^{j\omega})]^2}$$

Phase of $H(e^{j\omega})$ is called the phase response and it can be computed from the formula

$$\angle H(e^{j\omega}) = Tan^{-1}\left(\frac{H_I(e^{j\omega})}{H_R(e^{j\omega})}\right)$$

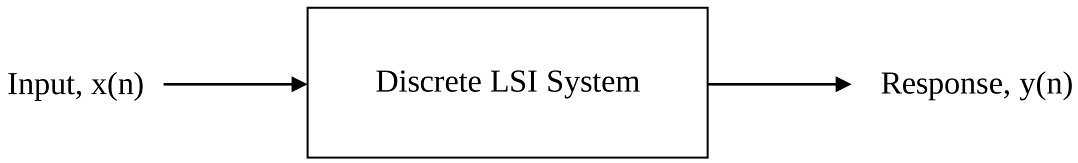
Impulse or Unit Sample Response:

Response of a discrete LSI system with an input of impulse or unit sample sequence is called is called impulse response or unit sample response and it is represented with h(n).

Unit sample response h(n) can be obtained from the frequency response $H(e^{j\omega})$ by using Inverse Discrete Time Fourier Transform (IDTFT).

$$h(n) = IDTFT\,[\,H(e^{j\omega})\,]$$

Response of Discrete LSI System:



Response of discrete LSI system y(n) can be obtained from the frequency domain by using Inverse Discrete Time Fourier Transform (IDTFT).

$$y(n) = IDTFT\,[\,Y(e^{j\omega})\,] = IDTFT\,[X(e^{j\omega})\,H(e^{j\omega})\,]$$

Example-1: Evaluate (a) Frequency Response (b) Magnitude Response and (c) Phase Response (d)Unit Sample Response of a discrete LSI system $y(n)-\frac{1}{4}y(n-2)=x(n)+\frac{1}{2}x(n-1)$.

(a)Frequency Response

Given discrete LSI system $y(n)-\frac{1}{4}y(n-2)=x(n)+\frac{1}{2}x(n-1)$

Apply DTFT

$$\Rightarrow DTFT\left[y(n)-\frac{1}{4}y(n-2)\right]=DTFT\left[x(n)+\frac{1}{2}x(n-1)\right]$$

$$\Rightarrow Y(e^{j\omega})-\frac{1}{4}e^{-j2\omega}Y(e^{j\omega})=X(e^{j\omega})+\frac{1}{2}e^{-j\omega}X(e^{j\omega})$$

$$\Rightarrow Y(e^{j\omega})\left(1-\frac{1}{4}e^{-j2\omega}\right)=X(e^{j\omega})\left(1+\frac{1}{2}e^{-j\omega}\right)$$

$$\Rightarrow \frac{Y(e^{j\omega})}{X(e^{j\omega})}=\frac{1+\frac{1}{2}e^{-j\omega}}{1-\frac{1}{4}e^{-j2\omega}}=\frac{1+\frac{1}{2}e^{-j\omega}}{\left(1+\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)}=\frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$H(e^{j\omega})=\frac{1}{1-\frac{1}{2}e^{-j\omega}}=\frac{2}{2-e^{-j\omega}}=\frac{2}{2-(Cos\omega-jSin\omega)}=\frac{2}{2-Cos\omega+jSin\omega}$$

(b)Magnitude Response

$$\left|H(e^{j\omega})\right|=\sqrt{[H_R(e^{j\omega})]^2+[H_I(e^{j\omega})]^2}$$

$$\left|H(e^{j\omega})\right|=\left|\frac{2}{2-Cos\omega+jSin\omega}\right|=\frac{2}{\sqrt{(2-Cos\omega)^2+(Sin\omega)^2}}=\frac{2}{\sqrt{4+Cos^2\omega-4Cos\omega+Sin^2\omega}}=\frac{2}{\sqrt{5-4Cos\omega}}$$

$$\left|H(e^{j\omega})\right|=\frac{2}{\sqrt{5-4Cos\omega}}$$

(c)Phase Response

$$\angle H(e^{j\omega}) = Tan^{-1}\left(\frac{H_I(e^{j\omega})}{H_R(e^{j\omega})}\right) = \angle(2+j0) - \angle(2-Cos\omega + jSin\omega) = Tan^{-1}\left(\frac{0}{2}\right) - Tan^{-1}\left(\frac{Sin\omega}{2-Cos\omega}\right)$$

$$\angle H(e^{j\omega}) = -Tan^{-1}\left(\frac{Sin\omega}{2-Cos\omega}\right)$$

(d)Unit Sample Response

$$h(n) = IDTFT[H(e^{j\omega})] = IDTFT\left(\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right) = \left(\frac{1}{2}\right)^n u(n)$$

Example-2: Evaluate the response of a discrete LSI system $y(n) - \frac{1}{4}y(n-2) = x(n) + \frac{1}{2}x(n-1)$ with an input of $x(n) = \left(\frac{1}{3}\right)^n u(n)$

Given discrete LSI system $y(n) - \frac{1}{4}y(n-2) = x(n) + \frac{1}{2}x(n-1)$

Apply DTFT

$$\Rightarrow DTFT\left[y(n) - \frac{1}{4}y(n-2)\right] = DTFT\left[x(n) + \frac{1}{2}x(n-1)\right]$$

$$\Rightarrow Y(e^{j\omega}) - \frac{1}{4}e^{-j2\omega}Y(e^{j\omega}) = X(e^{j\omega}) + \frac{1}{2}e^{-j\omega}X(e^{j\omega})$$

$$\Rightarrow Y(e^{j\omega})\left(1-\frac{1}{4}e^{-j2\omega}\right) = X(e^{j\omega})\left(1+\frac{1}{2}e^{-j\omega}\right)$$

$$\Rightarrow \frac{Y(e^{j\omega})}{X(e^{j\omega})} = H(e^{j\omega}) = \frac{1+\frac{1}{2}e^{-j\omega}}{1-\frac{1}{4}e^{-j2\omega}} = \frac{1+\frac{1}{2}e^{-j\omega}}{\left(1+\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}\right)} = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$\Rightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}),\; X(e^{j\omega}) = DTFT[x(n)] = DTFT\left(\left(\frac{1}{3}\right)^n u(n)\right) = \frac{1}{1-\frac{1}{3}e^{-j\omega}}$$

$$Y(e^{j\omega}) = \left(\frac{1}{1-\frac{1}{3}e^{-j\omega}}\right)\left(\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right) = \frac{-2}{1-\frac{1}{3}e^{-j\omega}} + \frac{3}{1-\frac{1}{2}e^{-j\omega}}$$

$$\Rightarrow y(n) = IDTFT\left(Y(e^{j\omega})\right) \Rightarrow y(n) = 3\left(\frac{1}{2}\right)^n u(n) - 2\left(\frac{1}{3}\right)^n u(n)$$

Descriptive Questions:

1. Draw the graphical representation and sequence form of a discrete time signal x(n)=r(n+2)-r(n-3)-5u(n-4) and also Evaluate the summation $\sum_{n=0}^{\infty} x(n)$,where r(n) is unit ramp sequence.

2. Draw the graphical representation of sequences (i) x(2n)+x(0.5n) (ii) 2x(3n-1)-3x(-2n+1), given that $x(n) = \{1,4,\underset{\uparrow}{3},2,5,6,7,8,9\}$.

3. Determine the convoluted signal y(n)=x(n)*h(n), given that $x(n)=a^n u(n)$ and $h(n)=b^n u(n)$, if (i) a≠b (ii) a=b.

4. Determine the convoluted sequence $y(n) = x_1(n) * x_2(n)$, where $x_1(n) = \{1,2,\underset{\uparrow}{3},4\}$ and $x_2(n) = \{5,\underset{\uparrow}{6},7,8,9\}$.

5. Apply convolution operation and obtain the functional representation of a sequence y(n)=x(n)*h(n), where, $h(n)= [1+h_1(n)]*h_2(n)$. Given that $h_1(n)=\alpha\delta(n-2)$ and $h_2(n)=\alpha^n\delta(n-3)$.

6. Analyze the following signals for causal, non-causal, bounded, un-bounded (i) $x_1(n)=(1/2)^n$ (ii) $x_2(n)=(2)^n$ (iii) $x_3(n)=(1/2)^n u(n)$ (iv) $x_4(n)=(2)^n u(n)$ (v) $x_5(n)=(1/2)^n u(-n-1)$ (v) $x_6(n)=(2)^n u(-n-1)$.

7. Analyze the following discrete time signals for periodic and aperiodic. If periodic, then compute the period.(i)x(n)=4Cos(nπ/4)+6Sin(nπ/6) (ii) y(n)=4Cos(n/4)+6Sin(n/6).

8. Analyze the following discrete time signals for even and odd. If fails to satisfy even and odd properties, then evaluate even and odd parts.(i) $x_1(n) = \{1,2,\underset{\uparrow}{3},1,2\}$ (ii) $x_2(n) = \{-1,-2,\underset{\uparrow}{3},1,2\}$.

9. Analyze the following signals for energy and power(i)$x(n)=(1/2)^n u(n)$ (ii)y(n)=u(n).

10. Solve the difference equation y(n)-ay(n-1)=x(n) to obtain the impulse response, hence obtain the system response for an input $x(n)=b^n u(n)$, where 0<a<1 and 0<b<1.

11. Analyze the following systems for linearity, shift invariance, causality and stability (i)y(n)=T{x(n)}=x(n^2) (ii) y(n)=T{x(n)}=x^2(n) (iii) $y(n)=T\{x(n)\}=\sum_{k=-\infty}^{|n|}x(k)$

12. Determine (a) DTFT of x(n) (b) Magnitude spectrum and (c)Phase spectrum.

    Given $x(n)=\{-1,0,1,\underset{\uparrow}{2},1,0,1,2,1,0,-1\}$.

13. Determine the DTFT of x(n), Magnitude spectrum and Phase spectrum for the signals (i)x(n)=$(1/2)^n$u(n) (ii)x(n)=$(2)^n$u(-n-1) (iii) $x(n)=\left(\frac{1}{2}\right)^{|n|}$.

14. Apply properties of DTFT and Evaluate (i) X($e^{j\omega}$) at ω = 0 (ii) $\int_{-\pi}^{\pi} X(e^{j\omega})\,d\omega$ (iii) $\int_{-\pi}^{\pi}\left|X(e^{j\omega})\right|^2 d\omega$ Given that DTFT[x(n)] = X($e^{j\omega}$) and $x(n)=\{1,4,\underset{\uparrow}{3},2,5,6,7,0,9\}$.

15. Apply properties and Evaluate (a)Frequency response (b)Magnitude response and (c)Phase response of a system having LCCDE y(n)-0.5y(n-1)=x(n)+0.5x(n-1).

16. Determine (a)Unit sample response (b)Unit step response and (c)Response of a causal system for an input x(n)=$(1/2)^n$u(n).Given LCCDE y(n)-0.25y(n-1)=x(n).

17. Find the response of the system for an input x(n)=$(1/4)^n$u(n) having LCCDE y(n)-5/6 y(n-1)+1/6 y(n-2)=x(n)+x(n-1). Given initial conditions y(-1)=y(-2)=1.

18. For the system y(n)-5/6 y(n-1)+1/6 y(n-2)=x(n)+x(n-1), find(a)Natural response for given initial conditions y(-1)=y(-2)=1 (b)Forced response for given input x(n)=$(1/2)^n$u(n) and (c)Response of the system for an input x(n)=$(1/2)^n$u(n) and initial conditions y(-1)=y(-2)=1.

19. Find the natural response of the system having LCCDE y(n)-2 y(n-1)+4 y(n-2)=x(n)+x(n-1) for given initial conditions y(-1)=y(-2)=1.

20. Analyze the following systems for causality and stability(a)y(n)-0.5y(n-1)=x(n) with zero initial conditions(b)y(n)-0.5y(n-1)=x(n) with initial condition y(-1)=1(c)y(n)-2y(n-1)=x(n) with zero initial conditions(d)y(n)-2y(n-1)=x(n) with initial condition y(-1)=1

Quiz Questions:

1.	Which of the following signal is even (1)u(t+3)-u(t-3) (2)u(n+3)-u(n-3) (3)u(t+3)-u(t-4) (4)u(n+3)-u(n-4)	1,4
2.	Relation between unit sample and unit stem signal is (1)δ(n)=u(n)-u(n-1) (2)δ(n)=u(n+1)-u(n-1) (3) $u(n)=\sum_{k=-\infty}^{n}\delta(k)$ (4) $u(n)=\sum_{k=0}^{\infty}\delta(n-k)$	1,3,4
3.	Convoluted sequence u(n)*u(n) is (1) u(n) (2) nu(n) (3) (n+1)u(n) (4) δ(n)	3
4.	Impulse response of the system having LCCDE y(n)-y(n-1)=x(n) (1) δ(n) (2) u(n+1) (3) u(n-1) (4) u(n)	4
5.	Total Energy under the signal x(n)=u(n) is (1) 0 (2) ∞ (3) 1 (4) 0.5	2
6.	Find the magnitude response of y(n) – ½ y(n – 1) = x(n) – ½ x(n – 1)	1
7.	Find the sequence x(2n), given x(n)={1,2,3}	{1,3}
8.	If x(n) = {1, 2, 3, 4, 6, 7, 8, 9} and DTFT[x(n)] = $X(e^{j\omega})$,then find $X(e^{j\omega})$ at ω = 0	40
9.	Match the following signals and operations (1) 2x(2n) (2) x(-n/2) (3) 2x(-n) (4) x(2n+3) a.Time shifting b.Time scaling c.Folding d.Amplitude scaling e.Time shifting	1-b,d 2-b,c 3-c,d 4-a,b
10.	Match the following systems and properties (1) y(n)=5 (2) y(n)=2x(n) (3) y(n)=x(2n) (4) $y(n)=x(n^2)$ a.Linear b.Shift Invariant c.Non-Linear d.Shift Variant	1-c,d 2-a,b 3-a,d 4-a,d